Pure powers in recurrence sequences

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Abstract. Let $G$ be a linear recursive sequence of order $k$ satisfying the recursion $G_n = A_1 G_{n-1} + \cdots + A_k G_{n-k}$. In the case $k=2$ it is known that there are only finitely many perfect powers in such a sequence.

Ribenboim and McDaniel proved for sequences with $k=2$, $G_0=0$ and $G_1=1$ that in general for a term $G_n$ there are only finitely many terms $G_m$ such that $G_n G_m$ is a perfect square. P. Kiss proved that for any $n$ there exists a number $q_0$, depending on $G$ and $n$, such that the equation $G_n G_x = w^q$ in positive integers $x, w, q$ has no solution with $x>n$ and $q>q_0$. We show that for any $n$ there are only finitely many $x_1, x_2, \ldots, x_k, x, w, q$ positive integers such that $G_n G_{x_1} \cdots G_{x_k} G_x = w^q$ and some conditions hold.

Let $R = R(A, B, R_0, R_1)$ be a second order linear recursive sequence defined by

$$R_n = AR_{n-1} + BR_{n-2} \quad (n > 1),$$

where $A$, $B$, $R_0$ and $R_1$ are fixed rational integers. In the sequel we assume that the sequence is not a degenerate one, i.e. $\alpha/\beta$ is not a root of unity, where $\alpha$ and $\beta$ denote the roots of the polynomial $x^2 - Ax - B$.

The special cases $R(1, 1, 0, 1)$ and $R(2, 1, 0, 1)$ of the sequence $R$ is called Fibonacci and Pell sequence, respectively.

Many results are known about the relationship of the sequences $R$ and perfect powers. For the Fibonacci sequence Cohn [2] and Wylie [23] showed that a Fibonacci number $F_n$ is a square only when $n = 0, 1, 2$ or 12. Pethő [12], furthermore London and Finkelstein [9,10] proved that $F_n$ is full cube only if $n = 0, 1, 2$ or 6. From a result of Ljunggren [8] it follows that a Pell number is a square only if $n = 0, 1$ or 7 and Pethő [12] showed that these are the only perfect powers in the Pell sequence. Similar, but more general results was showed by McDaniel and Ribenboim [11], Robbins [19,20] Cohn [3,4,5] and Pethő [15]. Shorey and Stewart [21] showed, that any non degenerate binary recurrence sequence contains only finitely many perfect powers which can be effectively determined. This results follows also from a result of Pethő [14].

* Research supported by the Hungarian National Research Science Foundation, Operating Grant Number OTKA T 16975 and 020295.
Another type of problems was studied by Ribenboim and McDaniel. For a sequence \( R \) we say that the terms \( R_m, R_n \) are in the same square-class if there exist non zero integers \( x, y \) such that
\[
R_m x^2 = R_n y^2,
\]
or equivalently
\[
R_m R_n = t^2,
\]
where \( t \) is a positive rational integer.

A square-class is called trivial if it contains only one element. Ribenboim [16] proved that in the Fibonacci sequence the square-class of a Fibonacci number \( F_m \) is trivial, if \( m \neq 1, 2, 3, 6 \) or 12 and for the Lucas sequence \( L(1, 1, 2, 1) \) the square-class of a Lucas number \( L_m \) is trivial if \( m \neq 0, 1, 3 \) or 6. For more general sequences \( R(A, B, 0, 1) \), with \( (A, B) = 1 \), Ribenboim and McDaniel [17] obtained that each square class is finite and its elements can be effectively computed (see also Ribenboim [18]).

Further on we shall study more general recursive sequences.

Let \( G = G(A_1, \ldots, A_k, G_0, \ldots, G_{k-1}) \) be a \( k \)th order linear recursive sequence of rational integers defined by
\[
G_n = A_1 G_{n-1} + A_2 G_{n-2} + \cdots + A_k G_{n-k} \quad (n > k - 1),
\]
where \( A_1, \ldots, A_k \) and \( G_0, \ldots, G_{k-1} \) are not all zero integers. Denote by \( \alpha = \alpha_1, \alpha_2, \ldots, \alpha_s \) the distinct zeros of the polynomial \( x^k - A_1 x^{k-1} - A_2 x^{k-2} - \cdots - A_k \). Assume that \( \alpha, \alpha_2, \ldots, \alpha_s \) has multiplicity 1, \( m_2, \ldots, m_s \) respectively and \( |\alpha| > |\alpha_i| \) for \( i = 2, \ldots, s \). In this case, as it is known, the terms of the sequence can be written in the form
\[
G_n = a \alpha^n + r_2(n) \alpha_2^n + \cdots + r_s(n) \alpha_s^n \quad (n \geq 0),
\]
where \( r_i(i = 2, \ldots, s) \) are polynomials of degree \( m_i - 1 \) and the coefficients of the polynomials and \( a \) are elements of the algebraic number field \( \mathbb{Q}(\alpha, \alpha_2, \ldots, \alpha_s) \). Shorey and Stewart [21] proved that the sequence \( G \) does not contain \( q \)th powers if \( q \) is large enough. This result follows also from [7] and [22], where more general theorems where showed.

Kiss [6] generalized the square-class notion of Ribenboim and McDaniel. For a sequence \( G \) we say that the terms \( G_m, G_n \) are in the same \( q \)th-power class if \( G_m G_n = w^q \), where \( w, q \) rational integers and \( q \geq 2 \).

In the above mentioned paper Kiss proved that for any term \( G_n \) of the sequence \( G \) there is no terms \( G_m \) such that \( m > n \) and \( G_n, G_m \) are elements of the same \( q \)th-power class if \( q \) sufficiently large.
The purpose of this paper is to generalize this result. We show that under certain conditions the number of the solutions of equation
\[ G_n G_{x_1} G_{x_2} \cdots G_{x_k} G_x = w^q \]
where \( n \) is fixed, are finite.

We use a well known result of Baker [1].

**Lemma.** Let \( \gamma_1, \ldots, \gamma_v \) be non-zero algebraic numbers. Let \( M_1, \ldots, M_v \) be upper bounds for the heights of \( \gamma_1, \ldots, \gamma_v \), respectively. We assume that \( M_v \) is at least 4. Further let \( b_1, \ldots, b_{v-1} \) be rational integers with absolute values at most \( B \) and let \( b_v \) be a non-zero rational integer with absolute value at most \( B' \). We assume that \( B' \) is at least three. Let \( L \) defined by
\[ L = b_1 \log \gamma_1 + \cdots + b_v \log \gamma_v, \]
where the logarithms are assumed to have their principal values. If \( L \neq 0 \), then
\[ |L| > \exp(-C(\log B' \log M_v + B/B')) \]
where \( C \) is an effectively computable positive number depending on only the numbers \( M_1, \ldots, M_{v-1}, \gamma_1, \ldots, \gamma_v \) and \( v \) (see Theorem 1 of [1] with \( \delta = 1/B' \)).

**Theorem.** Let \( G \) be a \( k \)th order linear recursive sequence satisfying the above conditions. Assume that \( a \neq 0 \) and \( G_i \neq a \alpha^i \) for \( i > n_0 \). Then for any positive integer \( n, k \) and \( K \) there exists a number \( q_0 \), depending on \( n, G, K \) and \( k \), such that the equation
\[ (2) \quad G_n G_{x_1} G_{x_2} \cdots G_{x_k} G_x = w^q \quad (n \leq x_1 \leq \cdots \leq x_k < x) \]
in positive integer \( x_1, x_2, \ldots, x_k, x, w, q \) has no solution with \( x_k < Kn \) and \( q > q_0 \).

**Proof of the theorem.** We can assume, without loss of generality, that the terms of the sequence \( G \) are positive. We can also suppose that \( n > n_0 \) and \( n \) sufficiently large since otherwise our result follows from [20] and [7].

Let \( x_1, x_2, \ldots, x_k, x, w, q \) positive integers satisfying (2) with the above conditions. Let \( \varepsilon_m \) be defined by
\[ \varepsilon_m := \frac{1}{a} r_2(m) \left( \frac{\alpha_2}{\alpha} \right)^m + \frac{1}{a} r_3(m) \left( \frac{\alpha_3}{\alpha} \right)^m + \cdots + \frac{1}{a} r_s(m) \left( \frac{\alpha_s}{\alpha} \right)^m \quad (m \geq 0). \]
By (1) we have

\[
(1 + \varepsilon_n)(1 + \varepsilon_x) \prod_{i=1}^{k} (1 + \varepsilon_{x_i}) a^{k+2} \alpha^{n+x_1+\cdots+x_k} = w^q
\]

from which

\[
q \log w = (k + 2) \log a + \left( n + x + \sum_{i=1}^{k} x_i \right) \log \alpha + \log (1 + \varepsilon_n) + \log (1 + \varepsilon_x) + \sum_{i=1}^{k} \log (1 + \varepsilon_{x_i})
\]

follows. It is obvious that \( x < n + x + \sum_{i=1}^{k} x_i < (k + 2)x \). Using that \( \log |1 + \varepsilon_m| \) is bounded and \( \lim_{m \to \infty} \frac{1}{a_i(m)}(\frac{\alpha_i}{\alpha})^m = 0 \) \((i = 2, \ldots, s)\), we have

\[
c_1 \frac{x}{q} < \log w < c_2 \frac{x}{q}
\]

where \( c_1 \) and \( c_2 \) are constants.

Let \( L \) be defined by

\[
L := \left| \log \frac{w^q}{G_n G_{x_1} G_{x_2} \cdots G_{x_k} a \alpha^x} \right| = |\log (1 + \varepsilon_x)|.
\]

By the definition of \( \varepsilon_x \) and the properties of logarithm function there exists a constant \( c_3 \) that

\[
L < e^{-c_3 x}.
\]

On the other hand, by the Lemma with \( v = k + 4, M_{k+4} = w, B' = q \) and \( B = x \) we obtain the estimation

\[
L = \left| q \log w - \log G_n - \sum_{i=1}^{k} \log G_{x_i} - \log a - x \log \alpha \right| > e^{-C(\log q \log w + x/q)}
\]

where \( C \) depends on heights. By \( x_k < Kn \) heights depend on \( G_n, \ldots, G_{Kn} \), i.e. on \( n, K, k \) and on the parameters of the recurrence. By (4), (5) and (6) we have \( c_3 x < C(\log q \log w + x/q) < c_4 \log q \log w \), i.e.

\[
x < c_5 \log q \log w
\]
with some \(c_3, c_4, c_5\). Using (4) and (7) we get \(c_6q \log w < x < c_5 \log q \log w\), i.e. \(q < c_7 \log q\), where \(c_6\) and \(c_7\) are constants. But this inequality does not hold if \(q > q_0 = q_0(G, n, K, k)\), which proves the theorem.

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