Publ. Math. Debrecen
53 / 1-2 (1998), 149–161

Strong laws of large numbers for pairwise independent random variables with multidimensional indices

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Abstract. Pairwise independent random variables with multidimensional indices are studied. The Kolmogorov and the Marcinkiewicz strong laws of large numbers and Spitzer's theorem are proved in the case of exponent $r \leq 1$.

1. Introduction

Several papers are devoted to the study of the strong law of large numbers for non independent random variables (see e.g. RÉVÉSZ [14] and CSÖRGŐ, TANDORI and TOTIK [1]). ETEMADI [2] proved that the Kolmogorov strong law of large numbers holds for identically distributed and pairwise independent random variables. KRUGLOV [11] extended that result and obtained the Marcinkiewicz strong law of large numbers and Spitzer's theorem in the pairwise independent case if $r < 1$.

On the other hand, the strong law of large numbers has been extended to the case where the index set is the positive integer d-dimensional lattice points (see e.g. GUT [6], KLESOV [9] and [10], FAZEKAS [3]). Moreover, the assumption of identical distribution can also be weakened. Among others HU, MÓRICZ and TAYLOR [8], GUT [7] and FAZEKAS [4] used domination of distributions instead of identical distribution.

Mathematics Subject Classification: 60F15.
Key words and phrases: strong law of large numbers, pairwise independent random variables, multidimensional indices.
This research was supported by Hungarian Foundation for Scientific Researches under Grant No. OTKA–TO16933 and by Hungarian Ministry of Culture and Education under Grant No. 179/1995.

In this paper the Kolmogorov and the Marcinkiewicz strong laws (if $0 < r < 1$) are proved for pairwise independent identically distributed random variables with multidimensional indices. Spitzer's theorem is obtained for pairwise independent dominated random variables with multidimensional indices. Our theorems in sections 3 and 4 are extensions of Theorems 1 and 2 of Kruglov [11]. Some parts of our theorems have been proved in Etemadi [2]. In section 5 we prove Spitzer's theorem (for $r < 1$) for weakly mean dominated random variables without assuming any independence.

We remark that there is a huge literature of laws of large numbers for mixing sequences (see e.g. Rio [15]) and for Banach space valued random variables (see e.g. Fazekas [3], Nguyen Van Giang [12] for the multiindex case). However, in this paper we concentrate on real-valued pairwise independent random variables.

2. Notation and preliminary lemmas

Let $\boldsymbol{N}^d$ be the positive integer d-dimensional lattice points, where d is a positive integer. For $\boldsymbol{n}, \boldsymbol{m} \in \boldsymbol{N}^d$, $\boldsymbol{n} \leq \boldsymbol{m}$ is defined coordinatewise, $(\boldsymbol{n}, \boldsymbol{m}] = \prod_{i=1}^d (n_i, m_i]$ is a d-dimensional rectangle and $|\boldsymbol{n}| = \prod_{i=1}^d n_i$, where $\boldsymbol{n} = (n_1, \dots, n_d)$, $\boldsymbol{m} = (m_1, \dots, m_d)$. $\sum_{\boldsymbol{n}}$ will denote the summation for all $\boldsymbol{n} \in \boldsymbol{N}^d$. $\boldsymbol{1} = (1, \dots, 1) \in \boldsymbol{N}^d$. $I(A)$ denotes the indicator function of the set A. We shall assume that random variables (r.v.'s) $\{X_{\boldsymbol{n}}, \boldsymbol{n} \in \boldsymbol{N}^d\}$ are defined on the same probability space $(\Omega, \mathcal{A}, \boldsymbol{P})$. $\boldsymbol{E}$ stands for the expectation. The following notation will be used: $S_{\boldsymbol{n}} = \sum_{\boldsymbol{k} \leq \boldsymbol{n}} X_{\boldsymbol{k}}$, $X_{\boldsymbol{n}}^+ = \max\{0, X_{\boldsymbol{n}}\}$, $X_{\boldsymbol{n}}^- = \max\{0, -X_{\boldsymbol{n}}\}$. Obviously, $X_{\boldsymbol{n}} = X_{\boldsymbol{n}}^+ - X_{\boldsymbol{n}}^-$, $|X_{\boldsymbol{n}}| = X_{\boldsymbol{n}}^+ + X_{\boldsymbol{n}}^-$. Different constants will be denoted by the same letter c.

The following two lemmas are proved e.g. in Gut [6] and in Fazekas [3].

Lemma 2.1. *Let X be a r.v. For $r > 0$ the following statements are equivalent:*

1) $\boldsymbol{E}\left(|X|^r \left(\log^+ |X|\right)^{d-1}\right) < \infty$,

2) $\sum_{\boldsymbol{n}} |\boldsymbol{n}|^{\alpha r - 1} \boldsymbol{P}\left(|X| \geq |\boldsymbol{n}|^\alpha \varepsilon\right) < \infty$, *for any $\alpha > 0$, $\varepsilon > 0$.*

Proof. We show 1) $\Rightarrow$ 2) in a particular case because later we shall use the inequality obtained. Let $d(k) = \mathrm{Card}\left\{\boldsymbol{n} : \boldsymbol{n} \in \boldsymbol{N}^d, |\boldsymbol{n}| = k\right\}$ and

$M(x) = \sum_{k \le x} d(k)$ (here k is a positive integer and x is a positive real number). It is known that $M(x) \sim \text{const}.x \left(\log^+ x\right)^{d-1}$, where $\log^+$ denotes the positive part of function log. We have

$$\begin{aligned}\sum_{\boldsymbol{n}} \boldsymbol{P}\left(|X| \ge |\boldsymbol{n}|\right) &= \sum_{\boldsymbol{n}} \sum_{i \ge |\boldsymbol{n}|} \boldsymbol{P}\left(i \le |X| < i+1\right) \\ &= \sum_{i} M(i) \boldsymbol{P}\left(i \le |X| < i+1\right) \\ \le c \sum_{i} i \left(\log^+ i\right)^{d-1} \boldsymbol{P}\left(i \le |X| < i+1\right) &\le c\boldsymbol{E}\left(|X| \left(\log^+ |X|\right)^{d-1}\right).\end{aligned}$$

Lemma 2.2. *Let $\{X_{\boldsymbol{n}}, \boldsymbol{n} \in \boldsymbol{N}^d\}$ be a sequence of identically distributed (i.d.) r.v.'s, $0 < r < p \le 2$, $\varepsilon > 0$ and define $Y_{\boldsymbol{n}} = X_{\boldsymbol{n}} I\left\{|X_{\boldsymbol{n}}| \le \varepsilon |\boldsymbol{n}|^{1/r}\right\}$. If*

$$\boldsymbol{E}\left(|X_{\boldsymbol{1}}|^r \left(\log^+ |X_{\boldsymbol{1}}|\right)^{d-1}\right) < \infty,$$

then

$$\sum_{\boldsymbol{n}} \boldsymbol{E}\left||\boldsymbol{n}|^{-1/r} Y_{\boldsymbol{n}}\right|^p < \infty.$$

Several classical theorems for identically distributed random variables remain valid for non identically distributed case if an appropriate domination condition is assumed. We shall use the so called weak mean domination (see e.g. Gut [7]).

Definition 2.3. It is said that the sequence $\{X_{\boldsymbol{n}}, \boldsymbol{n} \in \boldsymbol{N}^d\}$ is weakly mean dominated by the r.v. X if, for some $0 < c < \infty$,

(WMD) $$\frac{1}{|\boldsymbol{n}|} \sum_{\boldsymbol{k} \le \boldsymbol{n}} \boldsymbol{P}\left(|X_{\boldsymbol{k}}| > x\right) \le c\boldsymbol{P}\left(|X| > x\right)$$

for all $\boldsymbol{n} \in \boldsymbol{N}^d$ and $x > 0$.

Besides truncation, we shall use the following operations on r.v.'s. Let Z be a r.v., $\lambda > 0$, then define

(2.1) $$Z(\lambda) = |Z| I\left\{|Z| > \lambda\right\}$$

and

$$Z^*(\lambda) = |Z| I\left\{|Z| \le \lambda\right\} + \lambda I\left\{|Z| > \lambda\right\}. \tag{2.2}$$

The following lemma is a variant of Lemma 2.1 of Gut [7]. (See also Lemma 2.7 of Fazekas [4]).

Lemma 2.4. *Let* $\{X_{\boldsymbol{n}}, \boldsymbol{n} \in \boldsymbol{N}^d\}$ *be weakly mean dominated by* X*. Let* $p > 0$, $\lambda > 0$, *and let* $X_{\boldsymbol{n}}(\lambda)$ *and* $X^*_{\boldsymbol{n}}(\lambda)$ *be defined according to* (2.1) *and* (2.2), *respectively. Then*

$$\frac{1}{|\boldsymbol{n}|} \sum_{\boldsymbol{k} \le \boldsymbol{n}} \boldsymbol{E}\left(X^*_{\boldsymbol{k}}(\lambda)\right)^p \le c\boldsymbol{E}\left(X^*(\lambda)\right)^p, \tag{2.3}$$

and

$$\frac{1}{|\boldsymbol{n}|} \sum_{\boldsymbol{k} \le \boldsymbol{n}} \boldsymbol{E}\left(X_{\boldsymbol{k}}(\lambda)\right)^p \le c\boldsymbol{E}\left(X(\lambda)\right)^p. \tag{2.4}$$

Proof. For a non-negative r.v. Y we have $\boldsymbol{E}Y^p = p\int_0^\infty y^{p-1}\boldsymbol{P}(Y > y)dy$. By this equality and condition (WMD) it follows that

$$\begin{aligned}
\frac{1}{|\boldsymbol{n}|} \sum_{\boldsymbol{k} \le \boldsymbol{n}} \boldsymbol{E}\left(X^*_{\boldsymbol{k}}(\lambda)\right)^p &= \frac{1}{|\boldsymbol{n}|} \sum_{\boldsymbol{k} \le \boldsymbol{n}} p \int_0^\infty y^{p-1} \boldsymbol{P}\left(X^*_{\boldsymbol{k}}(\lambda) > y\right) dy \\
&= p \int_0^\lambda y^{p-1} \frac{1}{|\boldsymbol{n}|} \sum_{\boldsymbol{k} \le \boldsymbol{n}} \boldsymbol{P}\left(|X_{\boldsymbol{k}}| > y\right) dy \\
&\le p \int_0^\lambda y^{p-1} c\boldsymbol{P}(|X| > y) dy = c\boldsymbol{E}\left(X^*(\lambda)\right)^p.
\end{aligned}$$

Therefore (2.3) is proved. Similarly, (2.4) follows, since

$$\begin{aligned}
\frac{1}{|\boldsymbol{n}|} \sum_{\boldsymbol{k} \le \boldsymbol{n}} \boldsymbol{E}\left(X_{\boldsymbol{k}}(\lambda)\right)^p &= p \int_0^\infty y^{p-1} \frac{1}{|\boldsymbol{n}|} \sum_{\boldsymbol{k} \le \boldsymbol{n}} \boldsymbol{P}\left(X_{\boldsymbol{k}}(\lambda) > y\right) dy \\
&= p \int_0^\lambda y^{p-1} \frac{1}{|\boldsymbol{n}|} \sum_{\boldsymbol{k} \le \boldsymbol{n}} \boldsymbol{P}\left(|X_{\boldsymbol{k}}| > \lambda\right) dy \\
&\quad + p \int_\lambda^\infty y^{p-1} \frac{1}{|\boldsymbol{n}|} \sum_{\boldsymbol{k} \le \boldsymbol{n}} \boldsymbol{P}\left(|X_{\boldsymbol{k}}| > y\right) dy
\end{aligned}$$

$$\leq p \int_0^\infty y^{p-1} c\boldsymbol{P}(X(\lambda) > y)dy = c\boldsymbol{E}(X(\lambda))^p.$$

This completes the proof of Lemma 2.4.

Lemma. *2.5. Let $\{X_{\boldsymbol{n}}, \boldsymbol{n} \in \boldsymbol{N}^d\}$ be a sequence of pairwise independent r.v.'s, and let $\{a_{\boldsymbol{n}}, \boldsymbol{n} \in \boldsymbol{N}^d\}$ be a sequence of positive numbers. If*

$$\left\{\frac{a_{\boldsymbol{n}-\boldsymbol{v}}}{a_{\boldsymbol{n}}} : \boldsymbol{n} \in \boldsymbol{N}^d, \boldsymbol{v} \in V\right\}$$

is a bounded set, where $V = \{\boldsymbol{v} = (v_1, \dots, v_d) : v_i \in \{0,1\}\}$ and

$$\frac{S_{\boldsymbol{n}}}{a_{\boldsymbol{n}}} \to 0 \quad \textit{almost surely (a.s.)} \quad \textit{as} \quad |\boldsymbol{n}| \to \infty,$$

then

$$\sum_{\boldsymbol{n}} \boldsymbol{P}\left(|X_{\boldsymbol{n}}| \geq a_{\boldsymbol{n}}\right) < \infty.$$

The lemma follows from the Borel–Cantelli lemma for pairwise independent events (see e.g. Petrov [13] p. 214) by taking d-dimensional differences.

Lemma 2.6 (Kronecker)**.** *Let $x_{\boldsymbol{n}}$ and $b_{\boldsymbol{n}}$ be non-negative numbers ($\boldsymbol{n} \in \boldsymbol{N}^d$). Suppose that $b_{\boldsymbol{m}} \leq b_{\boldsymbol{n}}$ if $\boldsymbol{m} \leq \boldsymbol{n}$ and $b_{\boldsymbol{n}} \to \infty$ if $|\boldsymbol{n}| \to \infty$. If $\sum_{\boldsymbol{n}} x_{\boldsymbol{n}}$ is finite then*

$$\frac{1}{b_{\boldsymbol{n}}} \sum_{\boldsymbol{k} \leq \boldsymbol{n}} b_{\boldsymbol{k}} x_{\boldsymbol{k}} \to 0 \quad \textit{as} \quad |\boldsymbol{n}| \to \infty.$$

The proof is the same as in the case $d = 1$.

3. A general a.s. convergence theorem

The following result is a generalization of Theorem 1 of Kruglov [11].

Theorem 3.1. *Let $\{X_{\boldsymbol{n}}, \boldsymbol{n} \in \boldsymbol{N}^d\}$ be a sequence of non-negative r.v.'s, and let $\{b_{\boldsymbol{n}}, \boldsymbol{n} \in \boldsymbol{N}^d\}$ be a bounded sequence of non-negative numbers, and $B_{\boldsymbol{n}} = \sum_{\boldsymbol{k} \leq \boldsymbol{n}} b_{\boldsymbol{k}}$. If*

$$\sum_{\boldsymbol{n}} \frac{1}{|\boldsymbol{n}|} \boldsymbol{P}\left(|S_{\boldsymbol{n}} - B_{\boldsymbol{n}}| > \varepsilon |\boldsymbol{n}|^{1/r}\right) < \infty \tag{3.1}$$

for every $\varepsilon > 0$*, where* $0 < r \le 1$*, then*

$$\frac{1}{|\boldsymbol{n}|^{1/r}}\left(S_{\boldsymbol{n}} - B_{\boldsymbol{n}}\right) \to 0 \quad \text{a.s.} \quad \text{as} \quad |\boldsymbol{n}| \to \infty. \tag{3.2}$$

Proof. Fix $\alpha > 1$, $\varepsilon > 0$, denote the integer part of α^{n_i} by k_{n_i} $(i = 1, \dots, d)$ and let $\boldsymbol{k_n} = (k_{n_1}, \dots, k_{n_d})$. It follows from the inequalities

$$\begin{aligned}\sum_{\boldsymbol{n}} \frac{|\boldsymbol{k_{n+1}} - \boldsymbol{k_n}|}{|\boldsymbol{k_{n+1}}|} &\min_{\boldsymbol{k} \in (\boldsymbol{k_n}, \boldsymbol{k_{n+1}}]} \boldsymbol{P}\left(|S_{\boldsymbol{k}} - B_{\boldsymbol{k}}| > \varepsilon |\boldsymbol{k}|^{1/r}\right) \\ &\le \sum_{\boldsymbol{n}} \sum_{\boldsymbol{k} \in (\boldsymbol{k_n}, \boldsymbol{k_{n+1}}]} \frac{1}{|\boldsymbol{k}|} \boldsymbol{P}\left(|S_{\boldsymbol{k}} - B_{\boldsymbol{k}}| > \varepsilon |\boldsymbol{k}|^{1/r}\right) \\ &\le \sum_{\boldsymbol{n}} \frac{1}{|\boldsymbol{n}|} \boldsymbol{P}\left(|S_{\boldsymbol{n}} - B_{\boldsymbol{n}}| > \varepsilon |\boldsymbol{n}|^{1/r}\right)\end{aligned}$$

and condition (3.1) that there exists a sequence $\boldsymbol{m_n} = (m_{n_1}, \dots, m_{n_d})$, $\alpha^{n_i} < m_{n_i} \le \alpha^{n_i+1}$ $(i = 1, \dots, d)$ such that the series

$$\sum_{\boldsymbol{n}} \boldsymbol{P}\left(|S_{\boldsymbol{m_n}} - B_{\boldsymbol{m_n}}| > \varepsilon |\boldsymbol{m_n}|^{1/r}\right) \tag{3.3}$$

converges. By the Borel–Cantelli lemma, convergence of the series (3.3) implies

$$\frac{1}{|\boldsymbol{m_n}|^{1/r}} |S_{\boldsymbol{m_n}} - B_{\boldsymbol{m_n}}| \le \varepsilon \tag{3.4}$$

except for finitely many values of $\boldsymbol{m_n}$ a.s. For any $\boldsymbol{t} \in \boldsymbol{N}^d$ there exists an index $\boldsymbol{n} \in \boldsymbol{N}^d$ such that $\boldsymbol{t} \in (\boldsymbol{m_n}, \boldsymbol{m_{n+1}}]$. By non-negativity of $X_{\boldsymbol{k}}$ we have

$$\begin{aligned}\frac{1}{|\boldsymbol{t}|^{1/r}}\left(B_{\boldsymbol{m_n}} - B_{\boldsymbol{t}}\right) + \frac{1}{|\boldsymbol{t}|^{1/r}}\left(S_{\boldsymbol{m_n}} - B_{\boldsymbol{m_n}}\right) &\le \frac{1}{|\boldsymbol{t}|^{1/r}}\left(S_{\boldsymbol{t}} - B_{\boldsymbol{t}}\right) \\ \le \frac{1}{|\boldsymbol{t}|^{1/r}}\left(S_{\boldsymbol{m_{n+1}}} - B_{\boldsymbol{m_{n+1}}}\right) &+ \frac{1}{|\boldsymbol{t}|^{1/r}}\left(B_{\boldsymbol{m_{n+1}}} - B_{\boldsymbol{t}}\right).\end{aligned} \tag{3.5}$$

Put $b = \sup_{\boldsymbol{n}} b_{\boldsymbol{n}}$ and observe that

$$\frac{1}{|\boldsymbol{t}|^{1/r}}\left(B_{\boldsymbol{t}} - B_{\boldsymbol{m_n}}\right) \le \left(\alpha^{2d} - 1\right) b \alpha^{(1-1/r)\sum_{i=1}^{d} n_i} \le \left(\alpha^{2d} - 1\right) b$$

and

$$\frac{1}{|\boldsymbol{t}|^{1/r}}\left(B_{\boldsymbol{m}_{\boldsymbol{n}+\mathbf{1}}} - B_{\boldsymbol{t}}\right) \le \left(\alpha^{2d} - 1\right) b,$$

as $0 < r \le 1$. In view of the inequality $|\boldsymbol{t}|^{-1/r} \le \alpha^{2d/r}|\boldsymbol{m}_{\boldsymbol{n}+\mathbf{1}}|^{-1/r}$, (3.4) and (3.5) give

$$\frac{1}{|\boldsymbol{t}|^{1/r}}|S_{\boldsymbol{t}} - B_{\boldsymbol{t}}| \le \varepsilon\alpha^{2d/r} + \left(\alpha^{2d} - 1\right) b$$

except for finitely many $\boldsymbol{t}$ a.s. For any $\delta > 0$ we can find an $\varepsilon > 0$ and an $\alpha > 1$ such that $\varepsilon\alpha^{2d/r} + \left(\alpha^{2d} - 1\right) b < \delta$. Therefore (3.2) is proved.

4. Kolmogorov's SLLN

The following result extends Theorem 2 of Kruglov [11] for multi-index case.

Theorem 4.1. *Let $\{X_{\boldsymbol{n}}, \boldsymbol{n} \in \boldsymbol{N}^d\}$ be a sequence of pairwise independent r.v.'s. Assume that*

1) $\sup_{\boldsymbol{n}} \boldsymbol{E}|X_{\boldsymbol{n}}| < \infty$,
2) *condition* (WMD) *is satisfied with a r.v. X such that* $\boldsymbol{E}\left(|X|\left(\log^+|X|\right)^{d-1}\right) < \infty$. *Then*

$$\frac{1}{|\boldsymbol{n}|}\left(S_{\boldsymbol{n}} - \boldsymbol{E}S_{\boldsymbol{n}}\right) \to 0 \quad \text{a.s.} \qquad \text{as} \quad |\boldsymbol{n}| \to \infty, \tag{4.1}$$

moreover, for any $\varepsilon > 0$

$$\sum_{\boldsymbol{n}} \frac{1}{|\boldsymbol{n}|}\boldsymbol{P}\big(|S_{\boldsymbol{n}} - \boldsymbol{E}S_{\boldsymbol{n}}| > \varepsilon|\boldsymbol{n}|\big) < \infty. \tag{4.2}$$

Proof. First we prove (4.2). Put $Y_{\boldsymbol{k}} = X_{\boldsymbol{k}} I\left\{|X_{\boldsymbol{k}}| \le |\boldsymbol{n}|\right\}$, $\boldsymbol{k} \le \boldsymbol{n}$, $T_{\boldsymbol{n}} = \sum_{\boldsymbol{k}\le\boldsymbol{n}} Y_{\boldsymbol{k}}$. Remark that pairwise independence of $X_{\boldsymbol{k}}$ implies that of $Y_{\boldsymbol{k}}$. Condition 2) and (2.3) in Lemma 2.4 (with $\lambda = |\boldsymbol{n}|$, $p = 2$) imply the inequality

$$\frac{1}{|\boldsymbol{n}|}\sum_{\boldsymbol{k}\le\boldsymbol{n}} \boldsymbol{E}Y_{\boldsymbol{k}}^2 \le c|\boldsymbol{n}|^2\boldsymbol{P}\left(|X| > |\boldsymbol{n}|\right) + c\boldsymbol{E}\left(X^2 I\left\{|X| \le |\boldsymbol{n}|\right\}\right).$$

Further, we have

$$\sum_{\boldsymbol{n}} \frac{1}{|\boldsymbol{n}|^3} \boldsymbol{D}^2 T_{\boldsymbol{n}} = \sum_{\boldsymbol{n}} \frac{1}{|\boldsymbol{n}|^3} \sum_{\boldsymbol{k} \le \boldsymbol{n}} \boldsymbol{D}^2 Y_{\boldsymbol{k}} \le \sum_{\boldsymbol{n}} \frac{1}{|\boldsymbol{n}|^3} \sum_{\boldsymbol{k} \le \boldsymbol{n}} \boldsymbol{E} Y_{\boldsymbol{k}}^2$$
$$\le c \sum_{\boldsymbol{n}} \boldsymbol{P}\left(|X| > |\boldsymbol{n}|\right) + c \sum_{\boldsymbol{n}} \frac{1}{|\boldsymbol{n}|^2} \boldsymbol{E}\left(X^2 I\left\{|X| \le |\boldsymbol{n}|\right\}\right).$$

Obviously

$$\boldsymbol{P}\left(|S_{\boldsymbol{n}} - \boldsymbol{E}T_{\boldsymbol{n}}| > \varepsilon|\boldsymbol{n}|\right) \le \boldsymbol{P}\left(|T_{\boldsymbol{n}} - \boldsymbol{E}T_{\boldsymbol{n}}| > \varepsilon|\boldsymbol{n}|\right) + \boldsymbol{P}\left(\bigcup_{\boldsymbol{k} \le \boldsymbol{n}} \left\{|X_{\boldsymbol{k}}| > |\boldsymbol{n}|\right\}\right).$$

Therefore, by the Chebyshev inequality, Lemma 2.1 and Lemma 2.2, we have

$$\sum_{\boldsymbol{n}} \frac{1}{|\boldsymbol{n}|} \boldsymbol{P}\left(|S_{\boldsymbol{n}} - \boldsymbol{E}T_{\boldsymbol{n}}| > \varepsilon|\boldsymbol{n}|\right)$$
$$\le \frac{1}{\varepsilon^2} \sum_{\boldsymbol{n}} \frac{1}{|\boldsymbol{n}|^3} \boldsymbol{D}^2 T_{\boldsymbol{n}} + \sum_{\boldsymbol{n}} \frac{1}{|\boldsymbol{n}|} \sum_{\boldsymbol{k} \le \boldsymbol{n}} \boldsymbol{P}\left(|X_{\boldsymbol{k}}| > |\boldsymbol{n}|\right)$$
$$\le c\left(1 + \frac{1}{\varepsilon^2}\right) \sum_{\boldsymbol{n}} \boldsymbol{P}\left(|X| > |\boldsymbol{n}|\right) + \frac{c}{\varepsilon^2} \sum_{\boldsymbol{n}} \frac{1}{|\boldsymbol{n}|^2} \boldsymbol{E}\left(X^2 I\left\{|X| \le |\boldsymbol{n}|\right\}\right) < \infty.$$

Hence (4.2) follows, since

$$\frac{1}{|\boldsymbol{n}|} |\boldsymbol{E}S_{\boldsymbol{n}} - \boldsymbol{E}T_{\boldsymbol{n}}| \le \frac{1}{|\boldsymbol{n}|} \sum_{\boldsymbol{k} \le \boldsymbol{n}} \boldsymbol{E}\Big(|X_{\boldsymbol{k}}| I\left\{|X_{\boldsymbol{k}}| > |\boldsymbol{n}|\right\}\Big)$$
$$\le c\boldsymbol{E}\Big(|X| I\left\{|X| > |\boldsymbol{n}|\right\}\Big) \to 0,$$

as $|\boldsymbol{n}| \to \infty$, by (2.4) in Lemma 2.4 (with $\lambda = |\boldsymbol{n}|$, $p = 1$). Remark that assumption 1) is not used to prove (4.2). Now we turn to (4.1). It follows from the equality $|X_{\boldsymbol{n}}| = X_{\boldsymbol{n}}^+ + X_{\boldsymbol{n}}^-$ and condition 2) that

$$\frac{1}{|\boldsymbol{n}|} \sum_{\boldsymbol{k} \le \boldsymbol{n}} \boldsymbol{P}\left(X_{\boldsymbol{n}}^{\pm} > x\right) \le c\boldsymbol{P}\left(|X| > x\right)$$

for all $\boldsymbol{n} \in \boldsymbol{N}^d$ and $x > 0$. Therefore (4.2) holds with $X_{\boldsymbol{k}}$ replaced by $X_{\boldsymbol{k}}^+$ and $X_{\boldsymbol{k}}^-$. By Theorem 3.1 it follows that

$$\frac{1}{|\boldsymbol{n}|} \sum_{\boldsymbol{k} \le \boldsymbol{n}} \left(X_{\boldsymbol{k}}^{\pm} - \boldsymbol{E} X_{\boldsymbol{k}}^{\pm} \right) \to 0 \quad \text{a.s.} \qquad \text{as} \quad |\boldsymbol{n}| \to \infty.$$

Therefore (4.1) is proved.

Now we generalize Corollary 1 of Kruglov [11].

Corollary 4.2. *Let $\{X_{\boldsymbol{n}}, \boldsymbol{n} \in \boldsymbol{N}^d\}$ be a sequence of pairwise independent i.d.r.v.'s. The following statements are equivalent:*

1) $\boldsymbol{E}|X_{\boldsymbol{1}}| \left(\log^+ |X_{\boldsymbol{1}}|\right)^{d-1} < \infty$,

2) $|\boldsymbol{n}|^{-1} S_{\boldsymbol{n}} \to c$ *a.s., where c is some constant,*

3) *for any $\varepsilon > 0$ and some $b \ge 0$*

$$\sum_{\boldsymbol{n}} \frac{1}{|\boldsymbol{n}|} \boldsymbol{P} \left(\left| \sum_{\boldsymbol{k} \le \boldsymbol{n}} \left(|X_{\boldsymbol{k}}| - b \right) \right| > \varepsilon |\boldsymbol{n}| \right) < \infty.$$

Proof. Theorem 4.1 implies 1) $\Rightarrow$ 2) and 1) $\Rightarrow$ 3) with $b = \boldsymbol{E}|X_{\boldsymbol{1}}|$. Implication 2) $\Rightarrow$ 1) is a consequence of Lemma 2.5 and Lemma 2.1. It remained to prove implication 3) $\Rightarrow$ 1). By Theorem 3.1 we have

$$\frac{1}{|\boldsymbol{n}|} \sum_{\boldsymbol{k} \le \boldsymbol{n}} |X_{\boldsymbol{k}}| \to b \quad \text{a.s.} \qquad \text{as} \quad |\boldsymbol{n}| \to \infty.$$

So implication 3) $\Rightarrow$ 1) follows from implication 2) $\Rightarrow$ 1).

5. The Marcinkiewicz SLLN without assuming independence

It is known, that the Marcinkiewicz strong law of large numbers holds for identically distributed r.v.'s with arbitrary dependence structure, if $0 < r < 1$ (see e.g. Petrov [13], Chapter IV, Theorem 16). Now, we shall prove Spitzer's theorem and the Marcinkiewicz SLLN for non-independent r.v.'s satisfying condition (WMD) if $0 < r < 1$. (We remark, that the following result is implicitly contained in the proof of Theorem 4.1 of Fazekas [4].)

Theorem 5.1. *Let $\{X_{\boldsymbol{n}}, \boldsymbol{n} \in \boldsymbol{N}^d\}$ be weakly mean dominated by X such that*

$$\boldsymbol{E}\left(|X|^r \left(\log^+ |X|\right)^{d-1}\right) < \infty,$$

where $0 < r < 1$. Then

$$\sum_{\boldsymbol{n}} \frac{1}{|\boldsymbol{n}|} \boldsymbol{P}\left(|S_{\boldsymbol{n}}| > \varepsilon |\boldsymbol{n}|^{1/r}\right) < \infty$$

for any $\varepsilon > 0$, and

$$\frac{S_{\boldsymbol{n}}}{|\boldsymbol{n}|^{1/r}} \to 0 \quad \text{a.s.} \qquad \text{as} \quad |\boldsymbol{n}| \to \infty.$$

Proof. Let $Y_{\boldsymbol{k}} = X_{\boldsymbol{k}} I\{|X_{\boldsymbol{k}}| \le |\boldsymbol{n}|^{1/r}\}$ for $\boldsymbol{k} \le \boldsymbol{n}$. Then

$$\sum_{\boldsymbol{n}} \frac{1}{|\boldsymbol{n}|} \boldsymbol{P}\left(|S_{\boldsymbol{n}}| > \varepsilon |\boldsymbol{n}|^{1/r}\right) \le \sum_{\boldsymbol{n}} \frac{1}{|\boldsymbol{n}|} \boldsymbol{P}\left(\max_{\boldsymbol{k} \le \boldsymbol{n}} |X_{\boldsymbol{k}}| > |\boldsymbol{n}|^{1/r}\right) + \sum_{\boldsymbol{n}} \frac{1}{|\boldsymbol{n}|} \boldsymbol{P}\Big(\Big|\sum_{\boldsymbol{k} \le \boldsymbol{n}} Y_{\boldsymbol{k}}\Big| > \varepsilon |\boldsymbol{n}|^{1/r}\Big).$$

By (WMD) and Lemma 2.1

$$\sum_{\boldsymbol{n}} \frac{1}{|\boldsymbol{n}|} \boldsymbol{P}\left(\max_{\boldsymbol{k} \le \boldsymbol{n}} |X_{\boldsymbol{k}}| > |\boldsymbol{n}|^{1/r}\right) \le \sum_{\boldsymbol{n}} \frac{1}{|\boldsymbol{n}|} \sum_{\boldsymbol{k} \le \boldsymbol{n}} \boldsymbol{P}\left(|X_{\boldsymbol{k}}| > |\boldsymbol{n}|^{1/r}\right) \le c \sum_{\boldsymbol{n}} \boldsymbol{P}\left(|X| > |\boldsymbol{n}|^{1/r}\right) < \infty.$$

Let $\delta > 0$ such that $r + \delta < 1$. Then by Markov's and c_p-inequalities and Lemma 2.4

$$\sum_{\boldsymbol{n}} \frac{1}{|\boldsymbol{n}|} \boldsymbol{P}\left(\left|\sum_{\boldsymbol{k} \le \boldsymbol{n}} Y_{\boldsymbol{k}}\right| > \varepsilon |\boldsymbol{n}|^{1/r}\right) \le \sum_{\boldsymbol{n}} \frac{1}{|\boldsymbol{n}|} \frac{1}{|\boldsymbol{n}|^{(r+\delta)/r}} \frac{1}{\varepsilon^{r+\delta}} \boldsymbol{E}\left|\sum_{\boldsymbol{k} \le \boldsymbol{n}} Y_{\boldsymbol{k}}\right|^{r+\delta}$$

$$\le \sum_{\boldsymbol{n}} \frac{1}{|\boldsymbol{n}|} \frac{1}{|\boldsymbol{n}|^{(r+\delta)/r}} \frac{1}{\varepsilon^{r+\delta}} \sum_{\boldsymbol{k} \le \boldsymbol{n}} \boldsymbol{E}\left|Y_{\boldsymbol{k}}\right|^{r+\delta} \le c \sum_{\boldsymbol{n}} |\boldsymbol{n}|^{-(r+\delta)/r} \boldsymbol{E}\left|X'\right|^{r+\delta},$$

where $X' = XI\{|X| \le |\boldsymbol{n}|^{1/r}\} + |\boldsymbol{n}|^{1/r} I\{|X| > |\boldsymbol{n}|^{1/r}\}$. It is easy to see that

$$\begin{aligned}\boldsymbol{E}|X'|^{r+\delta} &= \int_0^{|\boldsymbol{n}|^{(r+\delta)/r}} \boldsymbol{P}\left(|X|^{r+\delta} > x\right) dx \\ &= \int_0^1 |\boldsymbol{n}|^{(r+\delta)/r} s^{\delta/r} \frac{r+\delta}{r} \boldsymbol{P}\left(|X| > |\boldsymbol{n}|^{1/r} s^{1/r}\right) ds.\end{aligned}$$

Now let $0 < \varrho < \delta/r$ and $\varrho_0 = \varrho/(d-1)$ if $d > 1$. Then by the above inequalities and the proof of Lemma 2.1 we get:

$$\begin{aligned}\sum_{\boldsymbol{n}} \frac{1}{|\boldsymbol{n}|} \boldsymbol{P}\left(\left|\sum_{\boldsymbol{k}\le\boldsymbol{n}} Y_{\boldsymbol{k}}\right| > \varepsilon|\boldsymbol{n}|^{1/r}\right) &\le c\int_0^1 s^{\delta/r} \sum_{\boldsymbol{n}} \boldsymbol{P}\left(|X| > |\boldsymbol{n}|^{1/r} s^{1/r}\right) ds \\ &\le c\int_0^1 s^{\delta/r} \boldsymbol{E}\left(|X|^r s^{-1}\left(\log^+(|X|^r s^{-1})\right)^{d-1}\right) ds \\ &\le c\int_0^1 s^{\delta/r} \boldsymbol{E}\left(|X|^r s^{-1}\left(\log^+|X|^r + s^{-\varrho_0}\right)^{d-1}\right) ds \\ &\le c\int_0^1 s^{\delta/r-1-\varrho} \boldsymbol{E}\left(|X|^r\left(\log^+|X|\right)^{d-1}\right) ds < \infty.\end{aligned}$$

The following corollary is an extension of Theorem 3 of Kruglov [11].

Corollary 5.2. *Let $\{X_{\boldsymbol{n}}, \boldsymbol{n} \in \boldsymbol{N}^d\}$ be a sequence of pairwise independent i.d.r.v.'s and $0 < r < 1$. The following statements are equivalent:*

1) $\boldsymbol{E}|X_{\boldsymbol{1}}|^r \left(\log^+|X_{\boldsymbol{1}}|\right)^{d-1} < \infty$,

2) $\sum_{\boldsymbol{n}} |X_{\boldsymbol{n}}|/|\boldsymbol{n}|^{1/r}$ *is convergent a.s.,*

3) $|\boldsymbol{n}|^{-1/r} S_{\boldsymbol{n}} \to 0$ *a.s.,*

4) *for any* $\varepsilon > 0$

$$\sum_{\boldsymbol{n}} \frac{1}{|\boldsymbol{n}|} \boldsymbol{P}\left(\sum_{\boldsymbol{k}\le\boldsymbol{n}} |X_{\boldsymbol{k}}| > \varepsilon|\boldsymbol{n}|^{1/r}\right) < \infty.$$

Proof. First we prove implication 3) $\Rightarrow$ 1). By Lemma 2.5 we have

$$\sum_{\boldsymbol{n}} \boldsymbol{P}\left(|X_{\boldsymbol{1}}| \ge |\boldsymbol{n}|^{1/r}\right) < \infty.$$

So 1) is a consequence of Lemma 2.1. Implication 4) $\Rightarrow$ 3) follows from Theorem 3.1. Now we prove 1) $\Rightarrow$ 2). Let $Y_{\boldsymbol{n}} = X_{\boldsymbol{n}} I\{|X_{\boldsymbol{n}}| \le |\boldsymbol{n}|^{1/r}\}$. Then by Lemma 2.2 we have $\sum_{\boldsymbol{n}} \boldsymbol{E}\left||\boldsymbol{n}|^{-1/r} Y_{\boldsymbol{n}}\right| < \infty$.
That is $\sum_{\boldsymbol{n}} |\boldsymbol{n}|^{-1/r}|Y_{\boldsymbol{n}}|$ is integrable, so it is finite a.s. By Lemma 2.1 $\sum_{\boldsymbol{n}} \boldsymbol{P}(X_{\boldsymbol{n}} \ne Y_{\boldsymbol{n}}) < \infty$ therefore Borel–Cantelli lemma implies 2). Implication 2) $\Rightarrow$ 3) is a consequence of Lemma 2.6.

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(Received April 14, 1997; revised November 21, 1997)