

A GENERAL APPROACH TO STRONG LAWS OF LARGE NUMBERS FOR FIELDS OF RANDOM VARIABLES

By

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1. Introduction and Notation

There are several methods to obtain almost sure (a.s.) convergence results for random fields (see e.g. [14], [9], [8], [5] and the literature cited there). The aim of our paper is to present a general approach to obtain strong laws of large numbers (SLLN) for random fields. Our method is an extension of the one given in [6]. In [6] only random sequences (i.e. not fields) were considered.

The paper is organized as follows. Section 2 contains the main result (Theorem 3). Once a maximal inequality is known, Theorem 3 easily implies an SLLN and it helps to obtain appropriate normalizing constants in the SLLN. The remaining sections contain applications. In Section 3 an SLLN is presented for logarithmically weighted sums. We remark that such kind of SLLN's are useful to prove almost sure central limit theorems (see e.g. [3]). In Section 4 the case of fields with superadditive moment structure is studied. In Section 5 a Brunk–Prokhorov type SLLN is presented. Section 6 is devoted to mixingales.

In the following \mathbb{N}_0 and \mathbb{N} denote the set of nonnegative and positive integers, respectively. Let d be a fixed positive integer. Throughout the paper I, J, K, L, M and N denote elements of \mathbb{N}_0^d (in particular, elements of \mathbb{N}^d). If an element of \mathbb{N}_0^d (or \mathbb{N}^d) is denoted by a capital letter, then its coordinates are denoted by the lower case of the same letter, i.e. N always means the vector

$(n_1, \dots, n_d) \in \mathbb{N}_0^d$. We also use $\mathbf{1} = (1, \dots, 1) \in \mathbb{N}^d$ and $\mathbf{0} = (0, \dots, 0) \in \mathbb{N}_0^d$. In \mathbb{N}_0^d we consider the coordinate-wise partial ordering: $M \leq N$ means $m_i \leq n_i$, $i = 1, \dots, d$ ($M < N$ means $M \leq N$ and $N \neq M$). $N \rightarrow \infty$ is interpreted as $n_i \rightarrow \infty$, $i = 1, \dots, d$, $\lim_N a_N$ is meant in this sense. In \mathbb{N}_0^d the maximum is defined coordinate-wise (actually we shall use it only for rectangles). If $N = (n_1, \dots, n_d) \in \mathbb{N}_0^d$ then $\langle N \rangle = \prod_{i=1}^d n_i$.

A numerical sequence a_N , $N \in \mathbb{N}_0^d$, is called d -sequence. If a_N is a d -sequence then its difference sequence, i.e. the d -sequence b_N for which $\sum_{M \leq N} b_M = a_N$, $N \in \mathbb{N}^d$, will be denoted by Δa_N .

We shall say that a d -sequence a_N is of product type if $a_N = \prod_{i=1}^d a_{n_i}^{(i)}$, where $a_{n_i}^{(i)}$ ($n_i = 0, 1, 2, \dots$) is a (single) sequence for each $i = 1, \dots, d$. Our consideration will be confined to normalizing constants of product type: b_N will always denote $b_N = \prod_{i=1}^d b_{n_i}^{(i)}$, where $b_{n_i}^{(i)}$, $n_i = 0, 1, 2, \dots$, is a nondecreasing sequence of positive numbers for each $i = 1, \dots, d$. In this case we shall say that b_N is a positive nondecreasing d -sequence of product type. Moreover, if for each $i = 1, \dots, d$ the sequence $b_{n_i}^{(i)}$ is unbounded, then b_N is called positive, nondecreasing, unbounded d -sequence of product type.

The random field will be denoted by X_N , $N \in \mathbb{N}_0^d$. S_N is the partial sum: $S_N = \sum_{M \leq N} X_M$ for $N \in \mathbb{N}_0^d$. As X_N is a field with lattice indices we shall say that X_N , $N \in \mathbb{N}_0^d$, is a d -sequence of random variables (r.v.'s). Remark that a sum or a maximum over the empty set will be interpreted as zero (i.e. $\sum_{N \in \mathcal{H}} X_N = \max_{N \in \mathcal{H}} X_N = 0$ if $\mathcal{H} = \emptyset$). As usual, $\log^+(x) = \max\{1, \log(x)\}$, $x > 0$.

2. The Basic SLLN

The proposition and lemma below are useful for proving Theorem 3. Proposition 1 and its proof are straightforward generalizations of Theorem 1.1 and its proof in [6]. Note that there are several other ways to obtain maximal inequalities of this type: see for example [8].

PROPOSITION 1. (Hájek–Rényi type maximal inequality.) *Let $N \in \mathbb{N}^d$ be fixed. Let r be a positive real number, a_N be a nonnegative d -sequence.*

Suppose that b_M is a positive, nondecreasing d -sequence of product type. Then

$$\mathbb{E} \left\{ \max_{L \leq M} |S_L|^r \right\} \leq \sum_{L \leq M} a_L \quad \forall M \leq N$$

implies

$$\mathbb{E} \left\{ \max_{M \leq N} \left| \frac{S_M}{b_M} \right|^r \right\} \leq 4^d \sum_{M \leq N} \frac{a_M}{b_M^r}.$$

PROOF. Without loss of generality we can assume that $b_1 = 1$. Fix an $N \in \mathbb{N}^d$ and for a moment a real number $c > 1$. For $I = (i_1, \dots, i_d) \in \mathbb{N}_0^d$ let us define the set

$$\mathcal{A}_I = \{J \in \mathbb{N}^d : J \leq N \text{ and } c^{i_k} \leq b_{j_k}^{(k)} < c^{i_k+1}, \quad k = 1, \dots, d\}.$$

Now we can form

$$D_I = \sum_{J \in \mathcal{A}_I} a_J \text{ and } K = \max\{I : \mathcal{A}_I \neq \emptyset\},$$

where D_I as we mentioned above is considered to be zero if $\mathcal{A}_I = \emptyset$. Note that K is well defined because of product form of b_N . It is easy to see that each nonempty \mathcal{A}_I has a maximal element. Therefore if $\mathcal{A}_I \neq \emptyset$ let

$$M_I = \max\{J : J \in \mathcal{A}_I\}$$

otherwise set $M_I = \mathbf{0}$. Since $\bigcup_{I \leq K} \mathcal{A}_I$ covers the rectangle $\{M \in \mathbb{N}^d : M \leq \leq N\}$ so

$$\mathbb{E} \left\{ \max_{M \leq N} \left| \frac{S_M}{b_M} \right|^r \right\} \leq \sum_{J \leq K} \mathbb{E} \left\{ \max_{I \in \mathcal{A}_J} \left| \frac{S_I}{b_I} \right|^r \right\}.$$

By the definition of \mathcal{A}_I , M_I and D_I we get

$$\begin{aligned} & \sum_{J \leq K} \mathbb{E} \left\{ \max_{I \in \mathcal{A}_J} \left| \frac{S_I}{b_I} \right|^r \right\} \leq \sum_{J \leq K} \left\{ \prod_{m=1}^d c^{-rj_m} \right\} \mathbb{E} \left\{ \max_{I \in \mathcal{A}_J} |S_I|^r \right\} \leq \\ & \leq \sum_{J \leq K} \left\{ \prod_{m=1}^d c^{-rj_m} \right\} \mathbb{E} \left\{ \max_{I \leq M_J} |S_I|^r \right\} \leq \sum_{J \leq K} \left\{ \prod_{m=1}^d c^{-rj_m} \right\} \sum_{I \leq M_J} a_I \leq \\ & \leq \sum_{J \leq K} \left\{ \prod_{m=1}^d c^{-rj_m} \right\} \sum_{I \leq J} D_I \leq \sum_{I \leq K} D_I \sum_{I \leq J \leq K} \left\{ \prod_{m=1}^d c^{-rj_m} \right\} \leq \end{aligned}$$

$$\begin{aligned}
&\leq \sum_{I \leq K} D_I \prod_{m=1}^d \left\{ \sum_{j=i_m}^{k_m} c^{-rj} \right\} \leq \sum_{I \leq K} D_I \prod_{m=1}^d \frac{c^{-rim}}{1-c^{-r}} \leq \\
&\leq \left\{ \frac{c^r}{1-c^{-r}} \right\}^d \sum_{I \leq K} D_I \prod_{m=1}^d c^{-r(im+1)} \leq \\
&\leq \left\{ \frac{c^r}{1-c^{-r}} \right\}^d \sum_{I \leq K} \left\{ \sum_{J \in \mathcal{A}_I} a_J \right\} \prod_{m=1}^d c^{-r(im+1)} \leq \\
&\leq \left\{ \frac{c^r}{1-c^{-r}} \right\}^d \sum_{I \leq K} \sum_{J \in \mathcal{A}_I} \frac{a_J}{b_J^r} \leq \left\{ \frac{c^r}{1-c^{-r}} \right\}^d \sum_{J \leq N} \frac{a_J}{b_J^r}.
\end{aligned}$$

This proves the proposition because $\inf_{c>1} \frac{c^r}{1-c^{-r}} = 4$. ■

LEMMA 2. Let a_N be a nonnegative d -sequence and let b_N be a positive, nondecreasing, unbounded d -sequence of product type. Suppose that $\sum_N \frac{a_N}{b_N^r} < +\infty$ with a fixed real $r > 0$. Then there exists a positive, nondecreasing, unbounded d -sequence β_N of product type for which

$$\lim_N \frac{\beta_N}{b_N} = 0 \quad \text{and} \quad \sum_N \frac{a_N}{\beta_N^r} < +\infty.$$

PROOF. Clearly it is enough to prove for $r = 1$. In case of $d = 1$ one can find our proposition in [6, Lemma 2.2]. Let $d \geq 2$. Then

$$+\infty > \sum_N \frac{a_N}{b_N} = \sum_{n_1} \frac{1}{b_{n_1}^{(1)}} \sum_{n_2, \dots, n_d} \frac{a_N}{\prod_{m=2}^d b_{n_m}^{(m)}} = \sum_{n_1} \frac{1}{b_{n_1}^{(1)}} T_{n_1}$$

with $T_{n_1} = \sum_{n_2, \dots, n_d} \frac{a_N}{\prod_{m=2}^d b_{n_m}^{(m)}}$. Applying the above mentioned lemma of [6],

we get that there exists an unbounded, positive, nondecreasing sequence $\beta_n^{(1)}$ so that

$$\lim_n \frac{\beta_n^{(1)}}{b_n^{(1)}} = 0 \quad \text{and} \quad \sum_{n_1} \frac{1}{\beta_{n_1}^{(1)}} T_{n_1} < +\infty.$$

If we have already obtained $\beta_n^{(m)}$ for $m = 1, \dots, k$, $k < d$ then replacing in the above procedure b_N by $\prod_{m=1}^k \beta_{n_m}^{(m)} \prod_{m=k+1}^d b_{n_m}^{(m)}$ and coordinate 1 by

coordinate $k + 1$ we get an appropriate $\beta_n^{(k+1)}$. Finally, by setting $\beta_N = \prod_{m=1}^d \beta_{n_m}^{(m)}$, it obviously satisfies the requirements. ■

The following theorem is an extension of Theorem 2.1 of [6].

THEOREM 3. *Let a_N, b_N be nonnegative d -sequences and let $r > 0$. Suppose that b_N is a positive, nondecreasing, unbounded d -sequence of product type. Then*

$$\sum_N \frac{a_N}{b_N^r} < +\infty \quad \text{and} \quad \mathbb{E} \left\{ \max_{M \leq N} |S_M|^r \right\} \leq \sum_{M \leq N} a_M \quad \forall N \in \mathbb{N}^d$$

imply

$$\lim_N \frac{S_N}{b_N} = 0 \quad \text{a.s.}$$

PROOF. Let β_N be the sequence obtained in the previous lemma. According to Proposition 1:

$$\mathbb{E} \left\{ \max_{M \leq N} \left| \frac{S_M}{\beta_M} \right|^r \right\} \leq 4^d \sum_{M \leq N} \frac{a_M}{\beta_M^r} \quad \forall N \in \mathbb{N}^d.$$

Hence

$$\mathbb{E} \left\{ \sup_{n_d} \dots \sup_{n_1} \left| \frac{S_N}{\beta_N} \right|^r \right\} \leq 4^d \sum_N \frac{a_N}{\beta_N^r}.$$

Since

$$\sup_{n_d} \dots \sup_{n_1} \left| \frac{S_N}{\beta_N} \right|^r = \sup_N \left| \frac{S_N}{\beta_N} \right|^r$$

it follows from the foregoing that

$$\sup_N \left| \frac{S_N}{\beta_N} \right|^r < +\infty \quad \text{a.s.}$$

We have

$$\left| \frac{S_N}{b_N} \right| = \frac{\beta_N}{b_N} \left| \frac{S_N}{\beta_N} \right| \leq \frac{\beta_N}{b_N} \sup_K \left| \frac{S_K}{\beta_K} \right|.$$

This proves the theorem because $\lim_N \frac{\beta_N}{b_N} = 0$. ■

DEFINITION 4. A function g on $\mathbb{N}^d \times \mathbb{N}^d$ is said to be *superadditive* if $g(I, (j_1, \dots, j_{m-1}, k, j_{m+1}, \dots, j_d)) + g(i_1, \dots, i_{m-1}, k + 1, i_{m+1}, \dots, i_d, J)$

can be majorized by $g(I, J)$ for any $m = 1, \dots, d$ and for any $i_m \leq k < j_m$. A d -sequence of random variables is said to have r -th moment function of superadditive structure (MFSS) if

$$\mathbb{E} \left\{ \left| \sum_{I \leq K \leq J} X_K \right|^r \right\} \leq g(I, J)^\alpha \quad \forall I, J \in \mathbb{N}^d,$$

where g is superadditive on $\mathbb{N}^d \times \mathbb{N}^d$, $r > 0$ and $\alpha > 1$. Remark that the notion of r -th MFSS was used by Móricz in [11].

REMARK 5. Maximal inequalities play important role in proving SLLN's. We shall frequently use the following result of Móricz (see [9, Corollary 1] or [12, Theorem 7]):

Suppose that $r \geq 1$ and X_N has r -th MFSS. Then there is a constant $A_{r,\alpha,d}$ for which

$$\mathbb{E} \left\{ \max_{K \leq N} |S_K|^r \right\} \leq A_{r,\alpha,d} g(1, N)^\alpha \quad \forall N \in \mathbb{N}^d.$$

The reader can verify that the above proposition is true in the case of $0 < r < 1$, too. ■

3. Logarithmically Weighted Sums

Móri in [15, Theorem 1] proved that the sequence

$$\frac{1}{\log^+ n} \sum_{k=1}^n \frac{X_k}{k}$$

converges a.s. to zero under general assumptions. With their general method in [6] Fazekas and Klesov proved a special case of Móri's theorem. Now we extend this case to fields of random variables. Our method is a generalization that of [6]. Our Lemma 6 and Theorem 7 are extensions of Lemma 9.1 and Theorem 9.1 of [6], respectively. In the lemma below $[x]$, $x \geq 0$ denotes the integer part of x , i.e. $[x]$ is the largest integer for which $[x] \leq x$.

LEMMA 6. (a) Let $n \in \mathbb{N}$ and $0 < \beta < 1$. Then there is a constant $C_{d,\beta}$ depending only on d and β such that:

$$\sum_{m_1=1}^n \sum_{m_2=1}^{\left[\frac{n}{m_1}\right]} \cdots \sum_{m_d=1}^{\left[\frac{n}{m_1 m_2 \cdots m_{d-1}}\right]} \frac{1}{\langle M \rangle^{1-\beta}} \leq C_{d,\beta} n^\beta (\log^+ n)^{d-1}.$$

(b) Let $0 < \beta < 1$, $1 < \gamma < 2$, $I, M, J \in \mathbb{N}^d$, $I \leq M \leq J$. Then there is a constant $C_{d,\beta}$ depending only on d and β such that:

$$\sum_{I \leq M \leq J} \sum_{\substack{I \leq K \leq J \\ \langle K \rangle \leq \langle M \rangle}} \frac{1}{\langle M \rangle^{1+\beta}} \frac{1}{(\log^+ \langle M \rangle)^{d-1}} \frac{1}{\langle K \rangle^{1-\beta}} \leq C_{d,\beta} \left\{ \sum_{I \leq M \leq J} \frac{1}{\langle M \rangle} \right\}^\gamma.$$

PROOF. (a) The case $d = 1$ is well known from elementary analysis. We prove by induction on d . Suppose that the statement is true for $d = f$. Let $n \in \mathbb{N}$ and $0 < \beta < 1$. Then

$$\begin{aligned} & \sum_{m_1=1}^n \sum_{m_2=1}^{\left[\frac{n}{m_1}\right]} \cdots \sum_{m_{f+1}=1}^{\left[\frac{n}{m_1 m_2 \cdots m_f}\right]} \frac{1}{\langle M \rangle^{1-\beta}} = \\ & = \sum_{m_1=1}^n \frac{1}{m_1^{1-\beta}} \sum_{m_2=1}^{\left[\frac{n}{m_1}\right]} \cdots \sum_{m_{f+1}=1}^{\left[\frac{n}{m_1 m_2 \cdots m_f}\right]} \frac{1}{(m_2 \cdots m_f)^{1-\beta}}. \end{aligned}$$

Now applying the hypothesis for $\left[\frac{n}{m_1}\right]$ we get that the above expression is majorized by:

$$\begin{aligned} C_{f,\beta} \sum_{m_1=1}^n \frac{1}{m_1^{1-\beta}} \left[\frac{n}{m_1}\right]^\beta \left\{ \log^+ \left[\frac{n}{m_1}\right] \right\}^{f-1} & \leq C_{f,\beta} n^\beta (\log^+ n)^{f-1} \sum_{m_1=1}^n \frac{1}{m_1} \leq \\ & \leq C_{f,\beta} n^\beta (\log^+ n)^{f-1} C \log^+ n \end{aligned}$$

with certain $C > 0$ (here we used the fact $\left[\frac{1}{c} \left[\frac{a}{b}\right]\right] = \left[\frac{a}{bc}\right]$ for $a, b, c \in \mathbb{N}$).

(b) In case $\sum_{I \leq M \leq J} \frac{1}{\langle M \rangle} \leq 1$ we get that

$$\begin{aligned} & \sum_{I \leq M \leq J} \sum_{\substack{I \leq K \leq J \\ \langle K \rangle \leq \langle M \rangle}} \frac{1}{\langle M \rangle^{1+\beta} (\log^+ \langle M \rangle)^{d-1} \langle K \rangle^{1-\beta}} \leq \\ & \leq \sum_{I \leq M \leq J} \sum_{\substack{I \leq K \leq J \\ \langle K \rangle \leq \langle M \rangle}} \frac{1}{\langle M \rangle^{1+\beta} \langle K \rangle^{1-\beta}} \leq \\ & \leq \sum_{I \leq M \leq J} \sum_{\substack{I \leq K \leq J \\ \langle K \rangle \leq \langle M \rangle}} \frac{1}{\langle M \rangle \langle K \rangle} \leq \left\{ \sum_{I \leq M \leq J} \frac{1}{\langle M \rangle} \right\}^2 \leq \left\{ \sum_{I \leq M \leq J} \frac{1}{\langle M \rangle} \right\}^\gamma. \end{aligned}$$

In case $\sum_{I \leq M \leq J} \frac{1}{\langle M \rangle} > 1$ using part (a) and the simple fact, that

$$\sum_{m_1=1}^n \sum_{m_2=1}^{\lfloor \frac{n}{m_1} \rfloor} \cdots \sum_{m_d=1}^{\lfloor \frac{n}{m_1 m_2 \cdots m_{d-1}} \rfloor} \frac{1}{\langle M \rangle^{1-\beta}} = \sum_{\substack{M \in \mathbb{N}^d \\ \langle M \rangle \leq n}} \frac{1}{\langle M \rangle^{1-\beta}}$$

holds for all $n, d \in \mathbb{N}$, we get that

$$\begin{aligned} & \sum_{I \leq M \leq J} \frac{1}{\langle M \rangle^{1+\beta} (\log^+ \langle M \rangle)^{d-1}} \sum_{\substack{I \leq K \leq J \\ \langle K \rangle \leq \langle M \rangle}} \frac{1}{\langle K \rangle^{1-\beta}} \leq \\ & \leq C_{d,\beta} \sum_{I \leq M \leq J} \frac{1}{\langle M \rangle^{1+\beta} (\log^+ \langle M \rangle)^{d-1}} \langle M \rangle^\beta (\log^+ \langle M \rangle)^{d-1} = \\ & = C_{d,\beta} \sum_{I \leq M \leq J} \frac{1}{\langle M \rangle} \leq C_{d,\beta} \left\{ \sum_{I \leq M \leq J} \frac{1}{\langle M \rangle} \right\}^\gamma. \quad \blacksquare \end{aligned}$$

THEOREM 7. Let $X_N, N \in \mathbb{N}^d$, be a d -sequence of random variables and suppose that for some $C > 0, \beta > 0$

$$|\mathbb{E}(X_K X_L)| \leq C \left\{ \frac{\langle K \rangle}{\langle L \rangle} \right\}^\beta \frac{1}{(\log^+ \langle L \rangle)^{d-1}} \quad \text{if } \langle K \rangle \leq \langle L \rangle,$$

Then

$$\lim_N \frac{1}{\prod_{i=1}^d \log^+ n_i} \sum_{K \leq N} \frac{X_K}{\langle K \rangle} = 0 \text{ a.s.}$$

PROOF. Clearly it is enough to prove for $0 < \beta < 1$. Let $I, J \in \mathbb{N}^d, I \leq J$. Using the assumptions we get:

$$\begin{aligned} \mathbb{E} \left\{ \left| \sum_{I \leq K \leq J} \frac{X_K}{\langle K \rangle} \right|^2 \right\} &\leq 2 \sum_{I \leq L \leq J} \sum_{\substack{I \leq K \leq J \\ \langle \bar{K} \rangle \leq \langle L \rangle}} \frac{1}{\langle K \rangle \langle L \rangle} |\mathbb{E}(X_K X_L)| \leq \\ &\leq 2C \sum_{I \leq L \leq J} \sum_{\substack{I \leq K \leq J \\ \langle \bar{K} \rangle \leq \langle L \rangle}} \frac{1}{\langle K \rangle^{1-\beta} \langle L \rangle^{1+\beta} (\log^+ \langle L \rangle)^{d-1}}. \end{aligned}$$

Let $1 < \gamma < 2$. It follows from Lemma 6(b) that

$$\mathbb{E} \left\{ \left| \sum_{I \leq K \leq J} \frac{X_K}{\langle K \rangle} \right|^2 \right\} \leq D_{d,\beta} \left\{ \sum_{I \leq L \leq J} \frac{1}{\langle L \rangle} \right\}^\gamma,$$

where $D_{d,\beta} > 0$ depends only on d and β . Now, from Remark 5 we get that

$$\mathbb{E} \left\{ \max_{I \leq J} \left| \sum_{K \leq I} \frac{X_K}{\langle K \rangle} \right|^2 \right\} \leq C_{d,\beta,\gamma} \left\{ \sum_{K \leq J} \frac{1}{\langle K \rangle} \right\}^\gamma \quad \forall J,$$

where $C_{d,\beta,\gamma} > 0$ depends only on d, β and γ . From the Hölder inequality we have:

$$\mathbb{E} \left\{ \max_{I \leq J} \left| \sum_{K \leq I} \frac{X_K}{\langle K \rangle} \right|^{\frac{2}{\gamma}} \right\} \leq (C_{d,\beta,\gamma})^{\frac{1}{\gamma}} \sum_{K \leq J} \frac{1}{\langle K \rangle} \quad \forall J.$$

Now we can apply Theorem 3 because

$$\sum_N \frac{1}{(\prod_{m=1}^d \log n_m)^{\frac{2}{\gamma}}} \frac{1}{\langle N \rangle} < +\infty. \quad \blacksquare$$

Now we state some analogues of Theorem 7.

REMARK 8. Let X_N be an orthogonal d -sequence of random variables, $r > 0$ and $s > \frac{1+r}{2}$. Suppose that for some $C > 0$

$$\mathbb{E}(X_K^2) \leq C \langle K \rangle^r.$$

Then for any $\rho > 1$

$$\lim_N \frac{1}{\left\{ \prod_{i=1}^d \log^+ n_i \right\}^\rho} \sum_{K \leq N} \frac{X_K}{\langle K \rangle^s} = 0 \quad \text{a.s.}$$

For the proof one can use the d -multiple version of the Rademacher–Menšov inequality [9, Corollary 3a]. ■

REMARK 9. Let $0 < r < 1$ and $0 < s \leq \frac{2}{3-r}$. Suppose that for some $C > 0$

$$|\mathbb{E}(X_K X_L)| \leq \frac{C \langle K \rangle^{sr}}{\langle L \rangle^s (\log^+ \langle L \rangle)^{d-1}} \quad \text{if } \langle K \rangle \leq \langle L \rangle.$$

Then

$$\lim_N \frac{1}{\langle N \rangle^{1-s}} \sum_{K \leq N} \frac{X_K}{\langle K \rangle^s} = 0 \quad \text{a.s.}$$

The proof is similar to that of Theorem 7. ■

4. Sequences with superadditive moment structure

In this section we prove a Marcinkiewicz–Zygmund type SLLN for d -sequences with superadditive moment structure. Our Proposition 11 is a generalization of Theorem 8.1 of [6]. For the sake of completeness we start with a simple technical lemma on partial summation.

LEMMA 10. Let a_N, b_N be nonnegative d -sequences such that $b_N = \frac{1}{\langle N \rangle^\alpha}$ for some $\alpha > 0$. Then

$$\sum_N (-1)^d \Lambda_N \Delta b_{N+1} < +\infty$$

implies

$$\sum_N a_N b_N < +\infty,$$

where $\Lambda_N = \sum_{M \leq N} a_M$. ■

PROPOSITION 11. Let $r > 0, \alpha > 1$ and suppose that X_N has r -th MFSS and $\Delta g(\mathbf{1}, N)^\alpha$ is nonnegative for any $N \in \mathbb{N}^d$. Then for arbitrary $q > 0$

$$(I) \quad \sum_N \frac{g(\mathbf{1}, N)^\alpha}{\langle N \rangle^{1+\frac{r}{q}}} < +\infty$$

implies

$$\lim \frac{S_N}{\langle N \rangle^{\frac{1}{q}}} = 0 \quad a.s.$$

PROOF. Using Remark 5 we get for all $N \in \mathbb{N}^d$ that:

$$\mathbb{E} \left\{ \max_{M \leq N} |S_M|^r \right\} \leq A_{r,\alpha,d} g(\mathbf{1}, N)^\alpha.$$

Let us introduce the notation $b_N = \frac{1}{\langle N \rangle^{\frac{r}{q}}}$. Since $\prod_{m=1}^d \left\{ \frac{1}{n_m^{\frac{r}{q}}} - \frac{1}{(n_m+1)^{\frac{r}{q}}} \right\} \leq$

$\leq C \frac{1}{\langle N \rangle^{1+\frac{r}{q}}}$ for some $C > 0$, so (I) implies

$$\sum_N (-1)^d g(\mathbf{1}, N)^\alpha \Delta b_{N+1} < +\infty.$$

Finally, we apply Lemma 10 and Theorem 3 to obtain the result. ■

5. A Brunk-Prokhorov Type Theorem

Let (Ω, \mathcal{A}, P) be a probability space. Let X_N and \mathcal{A}_N be a d -sequence of random variables and be a d -sequence of σ -subalgebras of \mathcal{A} , respectively. We shall say that the pair (X_N, \mathcal{A}_N) has property (ex) if

$$(ex) \quad \mathbb{E}(\mathbb{E}(X_L | \mathcal{A}_M) | \mathcal{A}_N) = \mathbb{E}(X_L | \mathcal{A}_{\min(M,N)}) \quad L, M, N \in \mathbb{N}^d.$$

This property is widely used in the theory of multiindex martingales (see e.g. [5]). Let X_N be a d -sequence of random variables and \mathcal{A}_N a nondecreasing d -sequence of sub σ -algebras of \mathcal{A} . We say that X_N is a martingale difference if

$$X_N \text{ is measurable with respect to } \mathcal{A}_N, \quad N \in \mathbb{N}^d,$$

$$\mathbb{E}(X_1) = 0 \text{ and } \mathbb{E}(X_N | \mathcal{A}_M) = 0 \text{ if } M < N.$$

In this section we shall use the Doob and the Burkholder inequalities for d -sequences of random variables. For the sake of completeness we state and prove these inequalities in the lemma below.

LEMMA 12. (a) (Doob's L^p -inequality.) Let $p > 1$. Then for any martingale (X_N, \mathcal{A}_N) having property (ex) for arbitrary $N \in \mathbb{N}^d$

$$\mathbb{E} \left\{ \max_{M \leq N} |X_M|^p \right\} \leq \left\{ \frac{p}{p-1} \right\}^{pd} \mathbb{E}(|X_N|^p).$$

(b) (Burkholder's inequality) Let $p > 1$. Then there is a constant $D_{p,d}$ such that for any martingale difference X_N having property (ex)

$$\mathbb{E}(|S_N|^{2p}) \leq D_{p,d} \mathbb{E} \left(\left\{ \sum_{M \leq N} X_M^2 \right\}^p \right) \quad \forall N \in \mathbb{N}^d.$$

PROPOSITION 13. Let X_N be a martingale difference having property (ex) and $p \geq 1$. Suppose that $\sum_{M \leq N} \mathbb{E}(|X_M|^{2p}) \leq C \langle N \rangle^r$ for some $C > 0$ and $r < p + 1$. Then $\lim_N \frac{S_N}{\langle N \rangle} = 0$ a.s.

PROOF. From Burkholder's inequality (Lemma 12(b)) and Hölder's inequality

$$\begin{aligned} \mathbb{E}(|S_N|^{2p}) &\leq D_{2p,2} \mathbb{E} \left\{ \left\{ \sum_{M \leq N} X_M^2 \right\}^p \right\} \leq \\ &\leq D_{2p,2} \langle N \rangle^{p-1} \sum_{M \leq N} \mathbb{E}(|X_M|^{2p}) \leq D_{2p,2} \langle N \rangle^{p+r-1}. \end{aligned}$$

Thus, by Doob's inequality (Lemma 12(a)),

$$\mathbb{E} \left\{ \max_{M \leq N} |S_M|^{2p} \right\} \leq F_{2p,2} \sum_{M \leq N} \Delta \langle M \rangle^{p+r-1}$$

for some constant $F_{2p,2} > 0$. Now $\Delta \langle M \rangle^{p+r-1} \leq C \langle M \rangle^{p+r-2}$ and Theorem 3 implies the result. \blacksquare

PROPOSITION 14. Let X_N be a martingale difference having property (ex) and let $p \geq 1$. Suppose that $\mathbb{E}(|X_N|^{2p})$ is d -sequence of product type. Then

$$\sum_N \frac{\mathbb{E}(|X_N|^{2p})}{b_N^{2p}} \langle N \rangle^{p-1} < +\infty$$

implies $\lim_N \frac{S_N}{b_N} = 0$ a.s., provided that b_N is a nondecreasing, positive, unbounded d -sequence of product type and either $p = 1$ or $\frac{\langle N \rangle^\delta}{b_N}$ is nonincreasing for some $\delta > \frac{p-1}{2p}$.

PROOF. Applying Lemma 12(b), Hölder's inequality and Lemma 12(a) we get

$$\mathbb{E} \left\{ \max_{M \leq N} |S_M|^{2p} \right\} \leq C_{p,d} \langle N \rangle^{p-1} \sum_{M \leq N} \mathbb{E}(|X_M|^{2p})$$

for some $C_{p,d} > 0$. In case $p = 1$ our main theorem and the above inequality imply the result. Let $p > 1$. Introduce the notation $c_N = \langle N \rangle^{p-1} \sum_{M \leq N} \mathbb{E}(|X_M|^{2p})$. It is easy to see that

$$\begin{aligned} \Delta c_N &= \prod_{l=1}^d \left\{ n_l^{p-1} \sum_{k=1}^{n_l} a_k^{(l)} - (n_l - 1)^{p-1} \sum_{k=1}^{n_l-1} a_k^{(l)} \right\} = \\ &= \prod_{l=1}^d \left\{ n_l^{p-1} a_{n_l}^{(l)} + \left\{ n_l^{p-1} - (n_l - 1)^{p-1} \right\} \sum_{k=1}^{n_l-1} a_k^{(l)} \right\} \leq \\ &\leq \prod_{l=1}^d \left\{ n_l^{p-1} a_{n_l}^{(l)} + C n_l^{p-2} \sum_{k=1}^{n_l-1} a_k^{(l)} \right\} \end{aligned}$$

for some $C > 0$, where $\prod_{l=1}^d a_{n_l}^{(l)} = \mathbb{E}(|X_N|^{2p})$. Using the assumptions we get

$$\begin{aligned} &\sum_{m=1}^n \frac{m^{p-2}}{b_m^{(l)2p}} \sum_{k=1}^{m-1} a_k^{(l)} = \sum_{k=1}^{n-1} a_k^{(l)} \sum_{m=k+1}^n \frac{m^{p-2}}{b_m^{(l)2p}} \leq \\ &\leq \sum_{k=1}^n a_k^{(l)} \sum_{m=k}^{\infty} \frac{m^{p-2}}{b_m^{(l)2p}} = \sum_{k=1}^n a_k^{(l)} \sum_{m=k}^{\infty} \frac{1}{m^r} \frac{m^{p+r-2}}{b_m^{(l)2p}} \leq \\ &\leq \sum_{k=1}^n a_k^{(l)} \frac{k^{p+r-2}}{b_k^{(l)2p}} \sum_{m=k}^{\infty} \frac{1}{m^r} \leq \sum_{k=1}^n a_k^{(l)} \frac{k^{p+r-2}}{b_k^{(l)2p}} C k^{1-r} \end{aligned}$$

for some $r > 1, C_r > 0$ and for each $1 \leq l \leq d$. This means that $\sum_N \Delta \frac{c_N}{b_N^{2p}} <$

$< +\infty$, hence one can apply Theorem 3. ■

We remark that a similar proposition can be proved in a similar manner for d -sequences having maximal coefficient of correlation strictly smaller than 1. For this, one can use [14, Lemma 4] instead of Burkholder's inequality.

6. Mixingales

In this chapter we define multiindex L' mixingales and prove an SLLN for a special class of such random variables. Remark that the notion of L' mixingales was introduced by McLeish [10] and Andrews [1]. Let \mathbb{Z} denote the set of integers and let

$$\mathcal{E}_N = \left\{ M \in \mathbb{Z}^d : 0 \leq n_k - m_k \leq 1, k = 1, \dots, d \text{ and } \sum_{k=1}^d (n_k - m_k) \text{ is even} \right\},$$

$$\mathcal{O}_N = \left\{ M \in \mathbb{Z}^d : 0 \leq n_k - m_k \leq 1, k = 1, \dots, d \text{ and } \sum_{k=1}^d (n_k - m_k) \text{ is odd} \right\},$$

if $N \in \mathbb{Z}^d$.

DEFINITION 15. Let $r \geq 1$, (Ω, \mathcal{A}, P) be a probability space, X_N be a d -sequence of random variables with finite r -th moment, \mathcal{A}_N ($N \in \mathbb{Z}^d$) be a nondecreasing d -sequence of σ -subalgebras of \mathcal{A} . The pair (X_N, \mathcal{A}_M) ($N \in \mathbb{N}^d, M \in \mathbb{Z}^d$) is called L' -mixingale if

$$(a) \quad \|\mathbb{E}(X_N | \mathcal{A}_{N-M})\|_r \leq c_N \Psi_{-M} \quad \text{if } m_i \geq 0 \text{ for some } i = 1, \dots, d,$$

$$(b) \quad \|X_N - \mathbb{E}(X_N | \mathcal{A}_{N+M})\|_r \leq c_N \Psi_M \quad \text{if } M \geq 0,$$

where c_N ($N \in \mathbb{N}^d$), Ψ_N ($N \in \mathbb{Z}^d$) are d -sequences with $\Psi_N \rightarrow 0$ as $n_i \rightarrow -\infty$ for some $i = 1, \dots, d$, $\Psi_N \rightarrow 0$ as $n_i \rightarrow \infty$ for each $i = 1, \dots, d$, and there is a constant $C > 0$ for which

$$\Psi_M \leq C \Psi_N$$

for any $N \in \mathbb{Z}^d$ and $M \in \mathcal{E}_N \cup \mathcal{O}_N$.

The following lemma is a straightforward generalization of Lemma 1 and Lemma 2 of [7].

LEMMA 16. (a) Let $r \geq 2$ and (X_N, \mathcal{A}_M) ($N \in \mathbb{N}^d, M \in \mathbb{Z}^d$) be an L' mixingale, having property (ex). Then there exists an $F_{r,d} > 0$ such that

$$\left\| \max_{M \leq N} |S_M| \right\|_r \leq F_{r,d} \sum_{K \in \mathbb{Z}^d} \left\{ \sum_{M \leq N} \|X_M^{(K)}\|_r^2 \right\}^{\frac{1}{2}},$$

where $X_M^{(K)} = \Delta \mathbb{E}(X_M | \mathcal{A}_{M-K})$ and here the difference is taken according to the subscript of \mathcal{A} while the subscript of X remains fixed.

(b) Let $r \geq 2$ and (X_N, \mathcal{A}_M) ($N \in \mathbb{N}^d, M \in \mathbb{Z}^d$) be an L^r mixingale, having property (ex) such that $\sum_{K \in \mathbb{Z}^d} \Psi_K < +\infty$. Then

$$\left\| \max_{M \leq N} |S_M| \right\|_r \leq C_{r,d} \left\{ \sum_{M \leq N} c_M^2 \right\}^{\frac{1}{2}}$$

for some $C_{r,d}$.

PROOF. (a) Let $N, K \in \mathbb{N}^d$. Then

$$\begin{aligned} \sum_{-K \leq M \leq K} X_N^{(M)} &= \sum_{-K \leq M \leq K} \Delta \mathbb{E}(X_N | \mathcal{A}_{N-M}) = \\ &= \mathbb{E}(X_N | \mathcal{A}_{N+K}) + \sum_{L \in \mathcal{L}_K^+} \mathbb{E}(X_N | \mathcal{A}_{N+L}) + (-1) \sum_{L \in \mathcal{L}_K^-} \mathbb{E}(X_N | \mathcal{A}_{N+L}), \end{aligned}$$

where

$$\mathcal{L}_K^+ = \{L \in \mathbb{Z}^d : l_i = k_i \text{ if } i \notin I \text{ and } l_i = -(k_i + 1) \text{ if } i \in I, \text{ for some } I \subset \{1, \dots, d\}, \text{ with } I \neq \emptyset \text{ and } \text{card}(I) \text{ is even}\},$$

$$\mathcal{L}_K^- = \{L \in \mathbb{Z}^d : l_i = k_i \text{ if } i \notin I \text{ and } l_i = -(k_i + 1) \text{ if } i \in I, \text{ for some } I \subset \{1, \dots, d\}, \text{ with } \text{card}(I) \text{ is odd}\}.$$

By the definition of the L^r -mixingale, one can see that

$$\lim_K \left\{ \sum_{-K \leq M \leq K} X_N^{(M)} - \mathbb{E}(X_N | \mathcal{A}_{N+K}) \right\} = \mathbf{0} \quad \text{in } L^r$$

and so

$$\lim_K \left\{ \sum_{-K \leq M \leq K} X_N^{(M)} - X_N \right\} = \mathbf{0} \quad \text{in } L^r.$$

Hence, using the triangle inequality in L^r , we get

$$\left\| \max_{M \leq N} |S_M| \right\|_r = \left\| \max_{M \leq N} \left| \sum_{L \leq M} \sum_{K \in \mathbb{Z}^d} X_L^{(K)} \right| \right\|_r \leq$$

$$\leq \left\| \max_{M \leq N} \sum_{K \in Z^d} \left\| \sum_{L \leq M} X_L^{(K)} \right\|_r \right\| \leq \sum_{K \in Z^d} \left\| \max_{M \leq N} \left\| \sum_{L \leq M} X_L^{(K)} \right\|_r \right\| = (I).$$

Let $K \in \mathbb{N}^d$ be fixed. With the help of property (ex) it is easy to check that the pair (Z_M, \mathcal{F}_M) is martingale difference, where

$$Z_M = X_M^{(K)} \quad \text{and} \quad \mathcal{F}_M = \mathcal{A}_{M-K}.$$

Hence by Lemma 12 (a), (b) and by the triangle inequality in the space $L^{\frac{r}{2}}$, we have

$$\begin{aligned} (I) &\leq D_{r,d} \sum_{K \in Z^d} \left\| \sum_{L \leq N} X_L^{(K)} \right\|_r \leq F_{r,d} \sum_{K \in Z^d} \left\| \left\{ \sum_{L \leq N} |X_L^{(K)}|^2 \right\}^{\frac{1}{2}} \right\|_r = \\ &= F_{r,d} \sum_{K \in Z^d} \left\| \left\{ \sum_{L \leq N} |X_L^{(K)}|^2 \right\} \right\|_{\frac{r}{2}}^{\frac{1}{2}} \leq F_{r,d} \sum_{K \in Z^d} \left\{ \sum_{L \leq N} \left\| |X_L^{(K)}|^2 \right\|_{\frac{r}{2}} \right\}^{\frac{1}{2}}. \end{aligned}$$

(b) Let us consider $X_L^{(K)}$. If $k_m \geq 0$ for some $m = 1, \dots, d$ then

$$\|\Delta \mathbb{E}(X_L | \mathcal{A}_{L-K})\|_r \leq c_L 2^d C \Psi_{-K}.$$

Otherwise, if $k_m \leq -1$ for each $m = 1, \dots, d$, then by Definition 15,

$$\|\Delta \mathbb{E}(X_L | \mathcal{A}_{L-K})\|_r \leq \sum_{M \in \mathcal{G}_{L-K} \cup \mathcal{O}_{L-K}} \|X_L - \mathbb{E}(X_L | \mathcal{A}_M)\|_r \leq c_L 2^d C \Psi_{-K}.$$

Hence, by part (a),

$$\begin{aligned} \left\| \max_{M \leq N} |S_M| \right\|_r &\leq F_{r,d} \sum_{K \in Z^d} \left\{ \sum_{L \leq N} c_L^2 2^{2d} C^2 \Psi_{-K}^2 \right\}^{\frac{1}{2}} = \\ &= F_{r,d} 2^d C \left\{ \sum_{K \in Z^d} \Psi_K \right\} \left\{ \sum_{L \leq N} c_L^2 \right\}^{\frac{1}{2}}. \end{aligned}$$

PROPOSITION 17. Let $r \geq 2$ and (X_N, \mathcal{A}_M) ($N \in \mathbb{N}^d, M \in \mathbb{Z}^d$) be an L^r mixingale of property (ex). Then

$$\sum_{N \in \mathbb{Z}^d} \Psi_N < \infty \quad \text{and} \quad \sum_{N \in \mathbb{N}^d} \frac{1}{\langle N \rangle^{1+\frac{r}{q}}} \left\{ \sum_{M \leq N} c_M^2 \right\}^{\frac{r}{2}} < \infty$$

imply

$$\lim_N \frac{S_N}{\langle N \rangle^{\frac{1}{q}}} = 0 \quad \text{a.s.}$$

provided that the d -sequence c_N is of product type.

PROOF. Easy consequence of Proposition 11 and Lemma 16(b). ■

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