ALMOST SURE CENTRAL LIMIT THEOREMS
FOR m-DEPENDENT RANDOM FIELDS

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Dedicated to the memory of Professor Péter Kiss

Abstract. It is proved that the almost sure central limit theorem holds true for m-dependent random fields.

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1. Introduction

Let \( \mathbb{N} \) be the set of the positive integers and \( \mathbb{N}^d \) the positive integer \( d \)-dimensional lattice points, where \( d \) is a fixed positive integer. Denote \( \mathbb{R} \) the set of real numbers and \( \mathcal{B} \) the \( \sigma \)-algebra of Borel sets of \( \mathbb{R} \). Let \( \zeta_n, n \in \mathbb{N}^d \), be a multiindex sequence of random variables on the probability space \((\Omega, \mathcal{A}, P)\). Almost sure limit theorems in multiindex case state that

\[
\frac{1}{D_n} \sum_{k \leq n} d_k \delta_{\zeta_k(\omega)} \Rightarrow \mu, \text{ as } n \to \infty, \text{ for almost every } \omega \in \Omega.
\]

Here \( \delta_x \) is the unit mass at point \( x \), that is \( \delta_x : \mathcal{B} \to \mathbb{R}, \delta_x(B) = 1 \) if \( x \in B \) and \( \delta_x(B) = 0 \) if \( x \notin B \), moreover \( \Rightarrow \mu \) denotes weak convergence to the probability measure \( \mu \). Theorems of this type are not direct consequences of the corresponding theorems for ordinary sequences.

In this paper \( \mathbf{k} = (k_1, \ldots, k_d), \mathbf{n} = (n_1, \ldots, n_d), \ldots \in \mathbb{N}^d \). Relations \( \leq, \nless, \min, \to \) etc. are defined coordinatewise, i.e. \( \mathbf{n} \to \infty \) means that \( n_i \to \infty \) for all \( i \in \{1, \ldots, d\} \). Let \( |\mathbf{n}| = \prod_{i=1}^d n_i \) and \( |\log \mathbf{n}| = \prod_{i=1}^d \log_+ n_i \), where \( \log_+ x = \log x \) if \( x \geq e \) and \( \log_+ x = 1 \) if \( x < e \).

In the multiindex version of the classical almost sure limit theorem \( \zeta_n = \frac{1}{\sqrt{|\mathbf{n}|}} \sum_{k \leq \mathbf{n}} X_k \), where \( X_k, \mathbf{k} \in \mathbb{N}^d \), are independent identically distributed random variables with expectation \( \mathbb{E}X_k = 0 \) and variance \( \mathbb{D}^2X_k = 1 \), moreover \( d_k = \frac{1}{|\mathbf{k}|}, D_n = |\log \mathbf{n}| \), finally \( \mu \) is the standard normal distribution \( \mathcal{N}(0,1) \). (See [2] in multiindex case, while [1] and [3] for single index case.)
We shall prove a similar proposition, but in so-called \( m \)-dependent case. For this purpose we need the next known theorems and lemmas.

**Theorem 1.1.** Assume that for any pair \( \mathbf{h}, \mathbf{l} \in \mathbb{N}^d \), \( \mathbf{h} \leq \mathbf{l} \) there exists a random variable \( \zeta_{\mathbf{h}, \mathbf{l}} \) with the following properties. \( \zeta_{\mathbf{h}, \mathbf{l}} = 0 \) if \( \mathbf{h} = \mathbf{l} \). If \( \mathbf{k}, \mathbf{l} \in \mathbb{N}^d \), then for \( \mathbf{h} = \min\{\mathbf{k}, \mathbf{l}\} \) we suppose that the following pairs of random variables are independent: \( \zeta_{\mathbf{k}} \) and \( \zeta_{\mathbf{h}, \mathbf{l}} \); \( \zeta_{\mathbf{l}} \) and \( \zeta_{\mathbf{h}, \mathbf{k}} \); \( \zeta_{\mathbf{h}, \mathbf{k}} \) and \( \zeta_{\mathbf{h}, \mathbf{l}} \). Assume that there exist \( c > 0 \) and \( \mathbf{n}_0 \in \mathbb{N}^d \) such that \( E(\zeta_{\mathbf{1}} - \zeta_{\mathbf{h}, \mathbf{l}})^2 \leq c|h|/|l| \) for all \( \mathbf{n}_0 \leq \mathbf{h}, \mathbf{l} \in \mathbb{N}^d \).

Let \( 0 \leq d_{k}^{(i)} \leq c \log \frac{k+1}{k} \), assume that \( \sum_{k=1}^{\infty} d_k^{(i)} = \infty \) for \( i \in \{1, \ldots, d\} \). Let \( d_k = \prod_{i=1}^{d} d_k^{(i)} \) and \( D_n = \sum_{k \leq n} d_k \). Then for any probability distribution \( \mu \) the following two statements are equivalent

\[
\frac{1}{D_n} \sum_{k \leq n} d_k \delta_{\zeta_k(\omega)} \Rightarrow \mu, \text{ as } n \to \infty, \text{ for almost every } \omega \in \Omega; \]

\[
\frac{1}{D_n} \sum_{k \leq n} d_k \mu_{\zeta_k} \Rightarrow \mu, \text{ as } n \to \infty,
\]

where \( \mu_{\zeta_k} \) denotes the distribution of the \( \zeta_k \).

**Proof.** Choose in [2], Theorem 2.1 and Remark 2.2, \( B = \mathbb{R} \), \( g(x, y) = |x - y| \), \( c_n = n \) and \( \beta = 1 \).

Let \( X_n \), \( \mathbf{n} \in \mathbb{N}^d \), be a multiindex sequence of random variables on the probability space \((\Omega, \mathcal{A}, P)\). Suppose that \( EX_n = 0 \) and \( D^2 X_n < \infty \) for all \( \mathbf{n} \in \mathbb{N}^d \). Let \( ||\mathbf{n}|| = \max\{n_1, \ldots, n_d\} \) and \( d(V_1, V_2) = \inf\{||\mathbf{n} - \mathbf{m}|| : \mathbf{n} \in V_1, \mathbf{m} \in V_2\} \), where \( V_1, V_2 \subset \mathbb{N}^d \). Let \( \sigma(V) \), where \( V \subset \mathbb{N}^d \), be the smallest \( \sigma \)-algebra with respect to which \( \{X_n, \mathbf{n} \in V\} \) are measurable.

**Definition 1.2.** Let \( m \in \mathbb{N} \) be fixed. The random field \( \{X_n, \mathbf{n} \in \mathbb{N}^d\} \) is said to be \( m \)-dependent if the \( \sigma \)-algebras \( \sigma(V_1) \) and \( \sigma(V_2) \) are independent whenever \( d(V_1, V_2) > m \), \( V_1, V_2 \subset \mathbb{N}^d \).

In the following let \( S_n = \sum_{k \leq n} X_k \), \( B_n = D^2 S_n \), \( \zeta_n = S_n/\sqrt{B_n} \) and let \( \mu_{\zeta_n} \) denote the distribution of the random variable \( \zeta_n \).

**Lemma 1.3.** Let \( \{X_n, \mathbf{n} \in \mathbb{N}^d\} \) be an \( m \)-dependent random field, \( EX_n = 0 \), \( \mathbf{n} \in \mathbb{N}^d \). Assume that

\[
\text{(1.1) } \text{there exist } M, \delta \in \mathbb{R} \text{ such that } E|X_n|^{2+\delta} \leq M < \infty \text{ for all } \mathbf{n} \in \mathbb{N}^d,
\]

for some \( \delta \geq 0 \). Then there exists constant \( C_\delta > 0 \) such that

\[
E|S_n|^{2+\delta} \leq C_\delta |\mathbf{n}|^{2+\delta}.
\]
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For all \( n \in \mathbb{N}^d \).

**Proof.** See [4], Lemma 5.

**Lemma 1.4.** Let \( \mu, \mu_n, n \in \mathbb{N}^d \), be distributions with \( \mu_n \Rightarrow \mu \), as \( n \to \infty \). Let \( d_k, k \in \mathbb{N}^d \), be a nonidentically zero sequence of nonnegative real numbers. Assume that for each fixed \( n_0 \in \mathbb{N}^d \),

\[
\frac{1}{\sum_{k \leq n} d_k} \sum_{k \in A_{n_0}} d_k \to 0, \quad \text{as} \quad n \to \infty,
\]

where \( A_{n_0} = \{ k \in \mathbb{N}^d : k \leq n \text{ and } k \not\geq n_0 \} \). Then

\[
\frac{1}{\sum_{k \leq n} d_k} \sum_{k \leq n} d_k \mu_k \Rightarrow \mu, \quad \text{as} \quad n \to \infty.
\]

**Proof.** Let \( f : \mathbb{R} \to \mathbb{R} \) be a bounded and continuous function. Then for \( \varepsilon > 0 \) there exists \( n_\varepsilon \in \mathbb{N}^d \) such that for \( n \geq n_\varepsilon \)

\[
\left| \int f \, d\mu_n - \int f \, d\mu \right| < \frac{\varepsilon}{2} \quad \text{and} \quad \frac{1}{\sum_{k \leq n} d_k} \sum_{k \in A_{n_\varepsilon}} d_k < \frac{\varepsilon}{2K},
\]

where \( \left| \int f \, d\mu_n - \int f \, d\mu \right| \leq K < \infty \). Let \( \gamma_n = \sum_{k \leq n} d_k \mu_k / \sum_{k \leq n} d_k \). Then

\[
\left| \int f \, d\gamma_n - \int f \, d\mu \right| \leq \frac{1}{\sum_{k \leq n} d_k} \sum_{k \in A_{n_\varepsilon}} d_k \left| \int f \, d\mu_k - \int f \, d\mu \right| + \frac{1}{\sum_{k \leq n} d_k} \sum_{n_\varepsilon \leq k \leq n} d_k \left| \int f \, d\mu_k - \int f \, d\mu \right| < \varepsilon,
\]

which implies Lemma 1.4.

It is easy to see that the conditions of Lemma 1.4 are satisfied for \( d_k = \frac{1}{|k|} \).

The next proposition is a central limit theorem for \( m \)-dependent random fields.

**Theorem 1.5.** Let \( \{ X_n, n \in \mathbb{N}^d \} \) be an \( m \)-dependent random field, \( EX_n = 0 \), \( n \in \mathbb{N}^d \). Assume that (1.1) holds for some \( \delta > 0 \) and

(1.2) \( \text{there exist } \sigma > 0 \text{ and } n_\sigma \in \mathbb{N}^d \text{ such that } \frac{B_n}{|n|} \geq \sigma \text{ for all } n \geq n_\sigma. \)
Then
\[ \mu_{\zeta_n} \Rightarrow \mathcal{N}(0, 1) \text{ as } n \to \infty. \]

**Proof.** It is a simple corollary of [4], Theorem 1.

2. Results

**Theorem 2.1.** Let \( \{X_n, n \in \mathbb{N}^d\} \) be an \( m \)-dependent random field, \( EX_n = 0, n \in \mathbb{N}^d \). Suppose that (1.1) and (1.2) hold for some \( \delta \geq 0 \). Let \( 0 \leq d^{(i)}_k \leq c \log \frac{k+1}{k} \), assume that \( \sum_{k=1}^{\infty} d^{(i)}_k = \infty \) for \( i \in \{1, \ldots, d\} \). Let \( d_k = \prod_{i=1}^{d} d^{(i)}_k \) and \( D_n = \sum_{k \leq n} d_k \).

Then for any probability distribution \( \mu \) the following two statements are equivalent

\[ \frac{1}{D_n} \sum_{k \leq n} d_k \delta_{\zeta_k(\omega)} \Rightarrow \mu, \text{ as } n \to \infty, \text{ for almost every } \omega \in \Omega; \]

\[ \frac{1}{D_n} \sum_{k \leq n} d_k \mu_{\zeta_k} \Rightarrow \mu, \text{ as } n \to \infty. \]

**Proof.** Let \( h, l \in \mathbb{N}^d, h \leq l, m = (m, \ldots, m) \in \mathbb{N}^d, V_l = \{ t \in \mathbb{N}^d : t \leq l \}, V_{h,1} = \{ t \in \mathbb{N}^d : t \leq l \text{ and } t \not\leq h + m \}, \zeta_{h,1} = \frac{1}{\sqrt{B_l}} \sum_{t \in V_{h,1}} X_t \). Let us verify in this case the assumptions of Theorem 1.1.

(I) \( \zeta_{l,1} = 0 \) because \( V_{l,1} = \emptyset \).

(II) Let \( k, l \in \mathbb{N}^d \) and \( h = \min\{k, l\} \). Then

\( \zeta_k \) is \( \sigma(V_k) \)-measurable, \( \zeta_l \) is \( \sigma(V_l) \)-measurable,

\( \zeta_{h,1} \) is \( \sigma(V_{h,1}) \)-measurable if \( V_{h,1} \neq \emptyset \), otherwise \( \zeta_{h,1} = 0 \),

\( \zeta_{h,k} \) is \( \sigma(V_{h,k}) \)-measurable if \( V_{h,k} \neq \emptyset \), otherwise \( \zeta_{h,k} = 0 \),

\[ d(V_k, V_{h,1}) > m \text{ if } V_{h,1} \neq \emptyset, \]

\[ d(V_l, V_{h,k}) > m \text{ if } V_{h,k} \neq \emptyset, \]

\[ d(V_{h,k}, V_{h,1}) > m \text{ if } V_{h,k} \neq \emptyset \text{ and } V_{h,1} \neq \emptyset. \]

Thus the following pairs of random variables are independent: \( \zeta_k \) and \( \zeta_{h,1} \); \( \zeta_l \) and \( \zeta_{h,k} \); \( \zeta_{h,k} \) and \( \zeta_{h,1} \).
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(III) By Lyapunov’s inequality, $(E|\xi|^s)^{1/s} \leq (E|\xi|^t)^{1/t}$ if $0 < s \leq t$. (See it for example in [5].) Thus we have

$$ES_{h+m}^2 \leq (E|S_{h+m}|^{2+\delta})^{2/\delta}.$$ 

By Lemma 1.3,

$$ES_{h+m}^2 \leq \left(c_1|h + m|^{\frac{2+\delta}{2}}\right)^{2/\delta} = c_2|h + m|.$$ 

Let $h, l \in \mathbb{N}^d$ such that $\max\{m, n\sigma\} \leq h \leq l$. Then $m \leq h$ and (2.1) imply that

$$E(\zeta_l - \zeta_{h,1})^2 = E\left(\frac{1}{\sqrt{B_l}}S_{h+m}\right)^2 = \frac{1}{B_l}ES_{h+m}^2 \leq \frac{c_2}{B_l}|h + m|.$$ 

Since $l \geq n\sigma$ thus, by assumption (1.2), $\frac{1}{B_l} \leq \frac{1}{\sigma l}$. So (2.2) implies that

$$E(\zeta_l - \zeta_{h,1})^2 \leq \frac{c_2}{\sigma} \frac{|h + m|}{l} = c_3 \frac{d}{l} \sum_{i=1}^d (h_i + m) \leq 2^d c_3 \frac{|h|}{l} \leq c_4 |h|.$$ 

Therefore random variables $\zeta_l$ and $\zeta_{h,1}$ satisfy the conditions of Theorem 1.1, which implies Theorem 2.1.

**Theorem 2.2.** Let $\{X_n, n \in \mathbb{N}^d\}$ be an $m$-dependent random field, $EX_n = 0$, $n \in \mathbb{N}^d$. Assume that (1.1) and (1.2) hold for some $\delta > 0$. Then

$$\frac{1}{|\log n|} \sum_{k \leq n} \frac{1}{|k|} \delta_{\zeta_k(\omega)} \Rightarrow \mathcal{N}(0, 1), \text{ as } n \to \infty, \text{ for almost every } \omega \in \Omega.$$ 

**Proof.** Let $d_k^{(i)} = \frac{1}{i}$, $k \in \mathbb{N}$, $i \in \{1, \ldots, d\}$. The conditions of Theorem 2.1 are satisfied, because $2 \leq (1 + \frac{1}{k})^k$, so $\frac{1}{k} \leq \frac{1}{\log 2} \log \frac{k+1}{k}$, moreover $\sum_{k=1}^\infty \frac{1}{k} = \infty$. Then $d_k = \frac{1}{|k|}$ and

$$D_n = \sum_{k \leq n} \prod_{i=1}^d \frac{1}{k_i} = \prod_{i=1}^d n_i \sum_{k \leq n} \frac{1}{k_i} \sim \prod_{i=1}^d \log n_i \sim |\log n|,$$

where $a_n \sim b_n$ if $a_n/b_n \to 1$, as $n \to \infty$. By Theorem 1.5, $\mu_{\zeta_n} \Rightarrow \mathcal{N}(0, 1)$, as $n \to \infty$. Therefore Lemma 1.4 implies that

$$\frac{1}{D_n} \sum_{k \leq n} d_k \mu_{\zeta_k} = \frac{1}{\sum_{k \leq n} |k|} \sum_{k \leq n} \frac{1}{|k|} \mu_{\zeta_k} \Rightarrow \mathcal{N}(0, 1), \text{ as } n \to \infty.$$
Now using Theorem 2.1, we obtain

\[
\frac{1}{\sum_{k \leq n} \frac{1}{|k|}} \sum_{k \leq n} \frac{1}{|k|} \delta_{k}(\omega) \Rightarrow N(0, 1), \text{ as } n \to \infty, \text{ for almost every } \omega \in \Omega.
\]

This fact and (2.3) imply Theorem 2.2.

References


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