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ALMOST SURE CENTRAL LIMIT THEOREMS FOR m-DEPENDENT RANDOM FIELDS

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Dedicated to the memory of Professor Péter Kiss

Abstract. It is proved that the almost sure central limit theorem holds true for m-dependent random fields.

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1. Introduction

Let $\mathbf{N}$ be the set of the positive integers and $\mathbf{N}^d$ the positive integer d-dimensional lattice points, where d is a fixed positive integer. Denote $\mathbf{R}$ the set of real numbers and $\mathcal{B}$ the σ-algebra of Borel sets of $\mathbf{R}$. Let $\zeta_{\mathbf{n}}$, $\mathbf{n} \in \mathbf{N}^d$, be a multiindex sequence of random variables on the probability space $(\Omega, \mathcal{A}, \mathrm{P})$. Almost sure limit theorems in multiindex case state that

$$\frac{1}{D_{\mathbf{n}}} \sum_{\mathbf{k} \leq \mathbf{n}} d_{\mathbf{k}} \delta_{\zeta_{\mathbf{k}}(\omega)} \Rightarrow \mu, \text{ as } \mathbf{n} \to \infty, \text{ for almost every } \omega \in \Omega.$$

Here δ_x is the unit mass at point x, that is $\delta_x : \mathcal{B} \to \mathbf{R}$, $\delta_x(B) = 1$ if $x \in B$ and $\delta_x(B) = 0$ if $x \notin B$, moreover $\Rightarrow \mu$ denotes weak convergence to the probability measure μ. Theorems of this type are not direct consequences of the corresponding theorems for ordinary sequences.

In this paper $\mathbf{k} = (k_1, \ldots, k_d), \mathbf{n} = (n_1, \ldots, n_d), \ldots \in \mathbf{N}^d$. Relations $\leq$, $\not\leq$, min, $\to$ etc. are defined coordinatewise, i.e. $\mathbf{n} \to \infty$ means that $n_i \to \infty$ for all $i \in \{1, \ldots, d\}$. Let $|\mathbf{n}| = \prod_{i=1}^{d} n_i$ and $|\log \mathbf{n}| = \prod_{i=1}^{d} \log_+ n_i$, where $\log_+ x = \log x$ if $x \geq e$ and $\log_+ x = 1$ if $x < e$.

In the multiindex version of the classical almost sure limit theorem $\zeta_{\mathbf{n}} = \frac{1}{\sqrt{|\mathbf{n}|}} \sum_{\mathbf{k} \leq \mathbf{n}} X_{\mathbf{k}}$, where $X_{\mathbf{k}}$, $\mathbf{k} \in \mathbf{N}^d$, are independent identically distributed random variables with expectation $\mathrm{E}X_{\mathbf{k}} = 0$ and variance $\mathrm{D}^2 X_{\mathbf{k}} = 1$, moreover $d_{\mathbf{k}} = \frac{1}{|\mathbf{k}|}$, $D_{\mathbf{n}} = |\log \mathbf{n}|$, finally μ is the standard normal distribution $\mathcal{N}(0,1)$. (See [2] in multiindex case, while [1] and [3] for single index case.)

We shall prove a similar proposition, but in so-called m-dependent case. For this purpose we need the next known theorems and lemmas.

Theorem 1.1. *Assume that for any pair* $\mathbf{h}, \mathbf{l} \in \mathbf{N}^d$, $\mathbf{h} \le \mathbf{l}$ *there exists a random variable* $\zeta_{\mathbf{h},\mathbf{l}}$ *with the following properties.* $\zeta_{\mathbf{h},\mathbf{l}} = 0$ *if* $\mathbf{h} = \mathbf{l}$. *If* $\mathbf{k}, \mathbf{l} \in \mathbf{N}^d$, *then for* $\mathbf{h} = \min\{\mathbf{k}, \mathbf{l}\}$ *we suppose that the following pairs of random variables are independent:* $\zeta_{\mathbf{k}}$ *and* $\zeta_{\mathbf{h},\mathbf{l}}$; $\zeta_{\mathbf{l}}$ *and* $\zeta_{\mathbf{h},\mathbf{k}}$; $\zeta_{\mathbf{h},\mathbf{k}}$ *and* $\zeta_{\mathbf{h},\mathbf{l}}$. *Assume that there exist* $c > 0$ *and* $\mathbf{n}_0 \in \mathbf{N}^d$ *such that* $E(\zeta_{\mathbf{l}} - \zeta_{\mathbf{h},\mathbf{l}})^2 \le c|\mathbf{h}|/|\mathbf{l}|$ *for all* $\mathbf{n}_0 \le \mathbf{h} \le \mathbf{l}$, $\mathbf{h}, \mathbf{l} \in \mathbf{N}^d$.

Let $0 \le d_k^{(i)} \le c \log \frac{k+1}{k}$, *assume that* $\sum_{k=1}^{\infty} d_k^{(i)} = \infty$ *for* $i \in \{1, \ldots, d\}$. *Let* $d_{\mathbf{k}} = \prod_{i=1}^{d} d_{k_i}^{(i)}$ *and* $D_{\mathbf{n}} = \sum_{\mathbf{k} \le \mathbf{n}} d_{\mathbf{k}}$. *Then for any probability distribution* μ *the following two statements are equivalent*

$$\frac{1}{D_{\mathbf{n}}} \sum_{\mathbf{k} \le \mathbf{n}} d_{\mathbf{k}} \delta_{\zeta_{\mathbf{k}}(\omega)} \Rightarrow \mu, \text{ as } \mathbf{n} \to \infty, \text{ for almost every } \omega \in \Omega;$$

$$\frac{1}{D_{\mathbf{n}}} \sum_{\mathbf{k} \le \mathbf{n}} d_{\mathbf{k}} \mu_{\zeta_{\mathbf{k}}} \Rightarrow \mu, \text{ as } \mathbf{n} \to \infty,$$

where $\mu_{\zeta_{\mathbf{k}}}$ *denotes the distribution of the* $\zeta_{\mathbf{k}}$.

Proof. Choose in [2], Theorem 2.1 and Remark 2.2, $B = \mathbf{R}$, $\varrho(x, y) = |x - y|$, $c_n^{(i)} = n$ and $\beta = 1$.

Let $X_{\mathbf{n}}$, $\mathbf{n} \in \mathbf{N}^d$, be a multiindex sequence of random variables on the probability space $(\Omega, \mathcal{A}, \mathrm{P})$. Suppose that $\mathrm{E}X_{\mathbf{n}} = 0$ and $\mathrm{D}^2 X_{\mathbf{n}} < \infty$ for all $\mathbf{n} \in \mathbf{N}^d$. Let $||\mathbf{n}|| = \max\{n_1, \ldots, n_d\}$ and $d(V_1, V_2) = \inf\{||\mathbf{n} - \mathbf{m}|| : \mathbf{n} \in V_1, \mathbf{m} \in V_2\}$, where $V_1, V_2 \subset \mathbf{N}^d$. Let $\sigma(V)$, where $V \subset \mathbf{N}^d$, be the smallest σ-algebra with respect to which $\{X_{\mathbf{n}}, \mathbf{n} \in V\}$ are measurable.

Definition 1.2. Let $m \in \mathbf{N}$ be fixed. The random field $\{X_{\mathbf{n}}, \mathbf{n} \in \mathbf{N}^d\}$ is said to be m-dependent if the σ-algebras $\sigma(V_1)$ and $\sigma(V_2)$ are independent whenever $d(V_1, V_2) > m$, $V_1, V_2 \subset \mathbf{N}^d$.

In the following let $S_{\mathbf{n}} = \sum_{\mathbf{k} \le \mathbf{n}} X_{\mathbf{k}}$, $B_{\mathbf{n}} = \mathrm{D}^2 S_{\mathbf{n}}$, $\zeta_{\mathbf{n}} = S_{\mathbf{n}}/\sqrt{B_{\mathbf{n}}}$ and let $\mu_{\zeta_{\mathbf{n}}}$ denote the distribution of the random variable $\zeta_{\mathbf{n}}$.

Lemma 1.3. *Let* $\{X_{\mathbf{n}}, \mathbf{n} \in \mathbf{N}^d\}$ *be an* m*-dependent random field,* $EX_{\mathbf{n}} = 0$, $\mathbf{n} \in \mathbf{N}^d$. *Assume that*

(1.1) *there exist* $M, \delta \in \mathbf{R}$ *such that* $E|X_{\mathbf{n}}|^{2+\delta} \le M < \infty$ *for all* $\mathbf{n} \in \mathbf{N}^d$,

for some $\delta \ge 0$. *Then there exists constant* $C_\delta > 0$ *such that*

$$E|S_{\mathbf{n}}|^{2+\delta} \le C_\delta |\mathbf{n}|^{\frac{2+\delta}{2}}$$

for all $\mathbf{n} \in \mathbf{N}^d$.

Proof. See [4], Lemma 5.

Lemma 1.4. *Let* $\mu, \mu_{\mathbf{n}}$, $\mathbf{n} \in \mathbf{N}^d$, *be distributions with* $\mu_{\mathbf{n}} \Rightarrow \mu$, *as* $\mathbf{n} \to \infty$. *Let* $d_{\mathbf{k}}$, $\mathbf{k} \in \mathbf{N}^d$, *be a nonidentically zero sequence of nonnegative real numbers. Assume that for each fixed* $\mathbf{n}_0 \in \mathbf{N}^d$,

$$\frac{1}{\sum\limits_{\mathbf{k}\le\mathbf{n}} d_{\mathbf{k}}} \sum_{\mathbf{k}\in A_{\mathbf{n}_0}} d_{\mathbf{k}} \to 0, \text{ as } \mathbf{n} \to \infty,$$

where $A_{\mathbf{n}_0} = \{\mathbf{k} \in \mathbf{N}^d : \mathbf{k} \le \mathbf{n} \text{ and } \mathbf{k} \not\ge \mathbf{n}_0\}$. *Then*

$$\frac{1}{\sum\limits_{\mathbf{k}\le\mathbf{n}} d_{\mathbf{k}}} \sum_{\mathbf{k}\le\mathbf{n}} d_{\mathbf{k}}\mu_{\mathbf{k}} \Rightarrow \mu, \text{ as } \mathbf{n} \to \infty.$$

Proof. Let $f\colon \mathbf{R} \to \mathbf{R}$ be a bounded and continuous function. Then for $\varepsilon > 0$ there exists $\mathbf{n}_\varepsilon \in \mathbf{N}^d$ such that for $\mathbf{n} \ge \mathbf{n}_\varepsilon$

$$\left|\int f\mathrm{d}\mu_{\mathbf{n}} - \int f\mathrm{d}\mu\right| < \frac{\varepsilon}{2} \quad \text{and} \quad \frac{1}{\sum\limits_{\mathbf{k}\le\mathbf{n}} d_{\mathbf{k}}} \sum_{\mathbf{k}\in A_{\mathbf{n}_\varepsilon}} d_{\mathbf{k}} < \frac{\varepsilon}{2K},$$

where $\left|\int f\mathrm{d}\mu_{\mathbf{n}} - \int f\mathrm{d}\mu\right| \le K < \infty$. Let $\gamma_{\mathbf{n}} = \sum\limits_{\mathbf{k}\le\mathbf{n}} d_{\mathbf{k}}\mu_{\mathbf{k}} \Big/ \sum\limits_{\mathbf{k}\le\mathbf{n}} d_{\mathbf{k}}$. Then

$$\begin{aligned}\left|\int f\mathrm{d}\gamma_{\mathbf{n}} - \int f\mathrm{d}\mu\right| &\le \frac{1}{\sum\limits_{\mathbf{k}\le\mathbf{n}} d_{\mathbf{k}}} \sum_{\mathbf{k}\in A_{\mathbf{n}_\varepsilon}} d_{\mathbf{k}} \left|\int f\mathrm{d}\mu_{\mathbf{k}} - \int f\mathrm{d}\mu\right| \\ &\quad + \frac{1}{\sum\limits_{\mathbf{k}\le\mathbf{n}} d_{\mathbf{k}}} \sum_{\mathbf{n}_\varepsilon\le\mathbf{k}\le\mathbf{n}} d_{\mathbf{k}} \left|\int f\mathrm{d}\mu_{\mathbf{k}} - \int f\mathrm{d}\mu\right| < \varepsilon,\end{aligned}$$

which implies Lemma 1.4.

It is easy to see that the conditions of Lemma 1.4 are satisfied for $d_{\mathbf{k}} = \frac{1}{|\mathbf{k}|}$. The next proposition is a central limit theorem for m-dependent random fields.

Theorem 1.5. *Let* $\{X_{\mathbf{n}}, \mathbf{n} \in \mathbf{N}^d\}$ *be an* m*-dependent random field,* $EX_{\mathbf{n}} = 0$, $\mathbf{n} \in \mathbf{N}^d$. *Assume that (1.1) holds for some* $\delta > 0$ *and*

(1.2) *there exist* $\sigma > 0$ *and* $\mathbf{n}_\sigma \in \mathbf{N}^d$ *such that* $\dfrac{B_{\mathbf{n}}}{|\mathbf{n}|} \ge \sigma$ *for all* $\mathbf{n} \ge \mathbf{n}_\sigma$.

Then

$$\mu_{\zeta_{\mathbf{n}}} \Rightarrow \mathcal{N}(0,1) \text{ as } \mathbf{n} \to \infty.$$

Proof. It is a simple corollary of [4], Theorem 1.

2. Results

Theorem 2.1. *Let $\{X_{\mathbf{n}}, \mathbf{n} \in \mathbf{N}^d\}$ be an m-dependent random field, $EX_{\mathbf{n}} = 0$, $\mathbf{n} \in \mathbf{N}^d$. Suppose that (1.1) and (1.2) hold for some $\delta \geq 0$. Let $0 \leq d_k^{(i)} \leq c \log \frac{k+1}{k}$, assume that $\sum_{k=1}^{\infty} d_k^{(i)} = \infty$ for $i \in \{1, \ldots, d\}$. Let $d_{\mathbf{k}} = \prod_{i=1}^{d} d_{k_i}^{(i)}$ and $D_{\mathbf{n}} = \sum_{\mathbf{k} \leq \mathbf{n}} d_{\mathbf{k}}$. Then for any probability distribution μ the following two statements are equivalent*

$$\frac{1}{D_{\mathbf{n}}} \sum_{\mathbf{k} \leq \mathbf{n}} d_{\mathbf{k}} \delta_{\zeta_{\mathbf{k}}(\omega)} \Rightarrow \mu, \text{ as } \mathbf{n} \to \infty, \text{ for almost every } \omega \in \Omega;$$

$$\frac{1}{D_{\mathbf{n}}} \sum_{\mathbf{k} \leq \mathbf{n}} d_{\mathbf{k}} \mu_{\zeta_{\mathbf{k}}} \Rightarrow \mu, \text{ as } \mathbf{n} \to \infty.$$

Proof. Let $\mathbf{h}, \mathbf{l} \in \mathbf{N}^d$, $\mathbf{h} \leq \mathbf{l}$, $\mathbf{m} = (m, \ldots, m) \in \mathbf{N}^d$, $V_{\mathbf{l}} = \{\mathbf{t} \in \mathbf{N}^d : \mathbf{t} \leq \mathbf{l}\}$, $V_{\mathbf{h},\mathbf{l}} = \{\mathbf{t} \in \mathbf{N}^d : \mathbf{t} \leq \mathbf{l} \text{ and } \mathbf{t} \not\leq \mathbf{h} + \mathbf{m}\}$, $\zeta_{\mathbf{h},\mathbf{l}} = \frac{1}{\sqrt{B_{\mathbf{l}}}} \sum_{\mathbf{t} \in V_{\mathbf{h},\mathbf{l}}} X_{\mathbf{t}}$. Let us verify in this case the assumptions of Theorem 1.1.

(I) $\zeta_{\mathbf{l},\mathbf{l}} = 0$ because $V_{\mathbf{l},\mathbf{l}} = \emptyset$.

(II) Let $\mathbf{k}, \mathbf{l} \in \mathbf{N}^d$ and $\mathbf{h} = \min\{\mathbf{k}, \mathbf{l}\}$. Then

$$\zeta_{\mathbf{k}} \text{ is } \sigma(V_{\mathbf{k}})\text{-measurable}, \quad \zeta_{\mathbf{l}} \text{ is } \sigma(V_{\mathbf{l}})\text{-measurable},$$

$$\zeta_{\mathbf{h},\mathbf{l}} \text{ is } \sigma(V_{\mathbf{h},\mathbf{l}})\text{-measurable if } V_{\mathbf{h},\mathbf{l}} \neq \emptyset, \text{ otherwise } \zeta_{\mathbf{h},\mathbf{l}} = 0,$$

$$\zeta_{\mathbf{h},\mathbf{k}} \text{ is } \sigma(V_{\mathbf{h},\mathbf{k}})\text{-measurable if } V_{\mathbf{h},\mathbf{k}} \neq \emptyset, \text{ otherwise } \zeta_{\mathbf{h},\mathbf{k}} = 0,$$

$$d(V_{\mathbf{k}}, V_{\mathbf{h},\mathbf{l}}) > m \text{ if } V_{\mathbf{h},\mathbf{l}} \neq \emptyset,$$

$$d(V_{\mathbf{l}}, V_{\mathbf{h},\mathbf{k}}) > m \text{ if } V_{\mathbf{h},\mathbf{k}} \neq \emptyset,$$

$$d(V_{\mathbf{h},\mathbf{k}}, V_{\mathbf{h},\mathbf{l}}) > m \text{ if } V_{\mathbf{h},\mathbf{k}} \neq \emptyset \text{ and } V_{\mathbf{h},\mathbf{l}} \neq \emptyset.$$

Thus the following pairs of random variables are independent: $\zeta_{\mathbf{k}}$ and $\zeta_{\mathbf{h},\mathbf{l}}$; $\zeta_{\mathbf{l}}$ and $\zeta_{\mathbf{h},\mathbf{k}}$; $\zeta_{\mathbf{h},\mathbf{k}}$ and $\zeta_{\mathbf{h},\mathbf{l}}$.

(III) By Lyapunov's inequality, $(\mathrm{E}|\xi|^s)^{1/s} \le (\mathrm{E}|\xi|^t)^{1/t}$ if $0 < s \le t$. (See it for example in [5].) Thus we have

$$\mathrm{E}S^2_{\mathbf{h}+\mathbf{m}} \le (\mathrm{E}|S_{\mathbf{h}+\mathbf{m}}|^{2+\delta})^{\frac{2}{2+\delta}}.$$

By Lemma 1.3,

$$\mathrm{E}S^2_{\mathbf{h}+\mathbf{m}} \le \left(c_1|\mathbf{h}+\mathbf{m}|^{\frac{2+\delta}{2}}\right)^{\frac{2}{2+\delta}} = c_2|\mathbf{h}+\mathbf{m}|. \tag{2.1}$$

Let $\mathbf{h},\mathbf{l} \in \mathbf{N}^d$ such that $\max\{\mathbf{m},\mathbf{n}_\sigma\} \le \mathbf{h} \le \mathbf{l}$. Then $\mathbf{m} \le \mathbf{h}$ and (2.1) imply that

$$\mathrm{E}(\zeta_{\mathbf{l}} - \zeta_{\mathbf{h},\mathbf{l}})^2 = \mathrm{E}\left(\frac{1}{\sqrt{B_{\mathbf{l}}}}S_{\mathbf{h}+\mathbf{m}}\right)^2 = \frac{1}{B_{\mathbf{l}}}\mathrm{E}S^2_{\mathbf{h}+\mathbf{m}} \le \frac{c_2}{B_{\mathbf{l}}}|\mathbf{h}+\mathbf{m}|. \tag{2.2}$$

Since $\mathbf{l} \ge \mathbf{n}_\sigma$ thus, by assumption (1.2), $\frac{1}{B_{\mathbf{l}}} \le \frac{1}{\sigma|\mathbf{l}|}$. So (2.2) implies that

$$\mathrm{E}(\zeta_{\mathbf{l}} - \zeta_{\mathbf{h},\mathbf{l}})^2 \le \frac{c_2}{\sigma}\frac{|\mathbf{h}+\mathbf{m}|}{|\mathbf{l}|} = c_3\frac{\prod_{i=1}^d (h_i + m)}{|\mathbf{l}|} \le 2^d c_3 \frac{|\mathbf{h}|}{|\mathbf{l}|} = c_4\frac{|\mathbf{h}|}{|\mathbf{l}|}.$$

Therefore random variables $\zeta_{\mathbf{l}}$ and $\zeta_{\mathbf{h},\mathbf{l}}$ satisfy the conditions of Theorem 1.1, which implies Theorem 2.1.

Theorem 2.2. *Let $\{X_{\mathbf{n}}, \mathbf{n} \in \mathbf{N}^d\}$ be an m-dependent random field, $EX_{\mathbf{n}} = 0$, $\mathbf{n} \in \mathbf{N}^d$. Assume that* (1.1) *and* (1.2) *hold for some $\delta > 0$. Then*

$$\frac{1}{|\log \mathbf{n}|}\sum_{\mathbf{k}\le\mathbf{n}} \frac{1}{|\mathbf{k}|}\delta_{\zeta_{\mathbf{k}}(\omega)} \Rightarrow \mathcal{N}(0,1), \text{ as } \mathbf{n} \to \infty, \text{ for almost every } \omega \in \Omega.$$

Proof. Let $d_k^{(i)} = \frac{1}{k}$, $k \in \mathbf{N}$, $i \in \{1,\dots,d\}$. The conditions of Theorem 2.1 are satisfied, because $2 \le \left(1+\frac{1}{k}\right)^k$, so $\frac{1}{k} \le \frac{1}{\log 2}\log\frac{k+1}{k}$, moreover $\sum_{k=1}^{\infty}\frac{1}{k} = \infty$. Then $d_{\mathbf{k}} = \frac{1}{|\mathbf{k}|}$ and

$$D_{\mathbf{n}} = \sum_{\mathbf{k}\le\mathbf{n}}\prod_{i=1}^d \frac{1}{k_i} = \prod_{i=1}^d\sum_{k_i=1}^{n_i}\frac{1}{k_i} \sim \prod_{i=1}^d \log n_i \sim |\log \mathbf{n}|, \tag{2.3}$$

where $a_{\mathbf{n}} \sim b_{\mathbf{n}}$ if $a_{\mathbf{n}}/b_{\mathbf{n}} \to 1$, as $\mathbf{n} \to \infty$. By Theorem 1.5, $\mu_{\zeta_{\mathbf{n}}} \Rightarrow \mathcal{N}(0,1)$, as $\mathbf{n} \to \infty$. Therefore Lemma 1.4 implies that

$$\frac{1}{D_{\mathbf{n}}}\sum_{\mathbf{k}\le\mathbf{n}} d_{\mathbf{k}}\mu_{\zeta_{\mathbf{k}}} = \frac{1}{\sum_{\mathbf{k}\le\mathbf{n}}\frac{1}{|\mathbf{k}|}}\sum_{\mathbf{k}\le\mathbf{n}}\frac{1}{|\mathbf{k}|}\mu_{\zeta_{\mathbf{k}}} \Rightarrow \mathcal{N}(0,1), \text{ as } \mathbf{n} \to \infty.$$

Now using Theorem 2.1, we obtain

$$\frac{1}{\sum\limits_{\mathbf{k}\le\mathbf{n}} \frac{1}{|\mathbf{k}|}} \sum_{\mathbf{k}\le\mathbf{n}} \frac{1}{|\mathbf{k}|}\delta_{\zeta_{\mathbf{k}}(\omega)} \Rightarrow \mathcal{N}(0,1), \text{ as } \mathbf{n}\to\infty, \text{ for almost every } \omega\in\Omega.$$

This fact and (2.3) imply Theorem 2.2.

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