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Convergence rates in the law of large numbers for arrays of Banach space valued random elements

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Abstract

A general convergence rate theorem is obtained for arrays of Banach space valued random elements. This theorem gives a unified approach to prove and extend several known results.

Keywords: Convergence rates; Complete convergence; Arrays of Banach space valued random variables

1. Introduction

Several papers are devoted to the study of convergence rates in the law of large numbers. The well-known theorem of Baum and Katz (1965) states the following. Let $X_1, X_2, \ldots$ be independent identically distributed random variables with $\boldsymbol{E}X_k = 0$ if $\boldsymbol{E}|X_k| < \infty$. Let $t > 0$, $r \geqslant 1$ and $2r > t$. Then $\boldsymbol{E}|X_k|^t < \infty$ if and only if

$$\sum_{n=1}^{\infty} n^{r-2} \boldsymbol{P}(|S_n| > \varepsilon n^{r/t}) < \infty \quad \text{for all} \quad \varepsilon > 0.$$

Earlier versions of this theorem were obtained by Hsu and Robbins (1947), Erdős (1949,1950) and Spitzer (1956). The result was extended to Banach space valued random variables (Jain, 1975; Woyczyński, 1980), to arrays of random variables (Hu et al., 1989; Gut, 1992). For the recent progress in this field see Ahmed et al. (2002) and Csörgő (2003).

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Instead of the whole sequence S_n, one can study the subsequence S_{k_n}. Most of the papers study subsequences by methods that are different from the ones used for the whole sequence. Similarly, papers dealing with arrays of random variables (see Gut, 1992; Fazekas, 1992; Hu et al., 1999) offer their own method to handle general arrays. The aim of this note is to show that an appropriate version of the classic result of Jain (1975) on S_n (Theorem 3.3) implies theorems on S_{k_n} for a broad class of k_n.

Throughout the paper we study Banach space valued random variables. However, some of our results are new for real variables, too. In Section 2 we introduce notation. The main results are in Section 3. Theorem 3.1 is a generalization of Theorem 3.3 of Jain (1975). The idea in Theorem 3.1 is the following. When we apply Hoffmann–Jørgensen's inequality, we use two different functions to obtain upper bounds for the two terms in the inequality. The theorem obtained seems to be difficult, but when we choose appropriate weight functions we can obtain several known theorems for general arrays like $X_{n1}, \ldots, X_{nk_n}$. Corollaries 3.2 and 3.3 are versions of Theorem 6.2 of Fazekas (1992) and Corollary 4.1 of Hu et al. (1999), respectively. In Section 4 we give the proofs. In Section 5 we specialize our result for Banach spaces which are of type p. Then we obtain new proofs for results in Fazekas (1992) and Hu et al. (1999).

2. Notation

Let $\mathbb{N}$ be the set of the positive integers, $\mathbb{R}$ the set of real numbers, $a \vee b = \max\{a, b\}$ and $a \wedge b = \min\{a, b\}$, where $a, b \in \mathbb{R}$. Denote by R_f the range of the function f and by $f \circ g$ the composite function of functions f and g.

Let Φ_0 denote the set of functions $f: [0, \infty) \to [0, \infty)$, that are nondecreasing. A function $f \in \Phi_0$ is said to satisfy the Δ_2-condition ($f \sim \Delta_2$) if there exists a constant $c > 0$, such that $f(2t) \leqslant cf(t)$ for all $t > 0$. It is clear that $f \sim \Delta_2$ iff for every fixed $k > 1$, there exists a constant $c > 1$, such that $f(kt) \leqslant cf(t)$ for all $t > 0$.

Throughout the paper let $\{k_n, n \in \mathbb{N}\}$ be a strictly increasing sequence of positive integers. Following Gut (1985), introduce the functions ψ and M_r with

$$\psi(t) = \text{Card}\{n \in \mathbb{N}: k_n \leqslant t\} \quad \text{for} \quad t > 0 \quad \text{and} \quad \psi(0) = 0,$$

and

$$M_r(t) = \sum_{i=1}^{[t]} k_i^{r-1} \quad \text{if} \quad t \geqslant 1 \quad \text{and} \quad M_r(t) = k_1^{r-1} \quad \text{if} \quad 0 \leqslant t < 1,$$

where $r \in \mathbb{R}$, Card A is the cardinality of the set A and $[.]$ denotes the integer function. Let $M = M_2$.

Let B be a real separable Banach space with norm $\|.\|$ and zero element $\mathbf{0}$. If X is a B-valued random variable (r.v.) and $\boldsymbol{E}\|X\| < \infty$ then $\boldsymbol{E}X$ stands for the Bochner integral of X.

X is *symmetric* if X and $-X$ have same distribution. The *symmetrization procedure* consists in assigning to the r.v. X the *symmetrized* r.v. $X^* = X - X'$, where X' is independent of X and has the same distribution. Then

$$\boldsymbol{P}(\|X'\|<t)\boldsymbol{P}(\|X\|>2t)\leqslant \boldsymbol{P}(\|X^*\|>t)\leqslant 2\boldsymbol{P}(\|X-b\|>t/2) \tag{2.1}$$

for all $t\geqslant 0$ and $b\in B$.

Let $\{X_{nk}, n\in\mathbb{N}, k=1,\ldots,k_n\}$ be an array of B-valued r.v.'s. It is rowwise independent, if $X_{n1},\ldots,X_{nk_n}$ are independent r.v.'s for any fixed $n\in\mathbb{N}$. Let $S_{k_n}=\sum_{k=1}^{k_n} X_{nk}$. If $k_n=n$ for all n, then we denote S_{k_n} by S_n. This corresponds to the case of ordinary sequences.

Definition 2.1 (Gut, 1992). We say that the array $\{X_{nk}, n\in\mathbb{N}, k=1,\ldots,k_n\}$ is *weakly mean dominated* (w.m.d.) by the r.v. X, if for some $\gamma>0$,

$$\frac{1}{k_n}\sum_{k=1}^{k_n}\boldsymbol{P}(\|X_{nk}\|>t)\leqslant\gamma\boldsymbol{P}(|X|>t)\quad\text{for all}\quad t\geqslant 0\ \text{and}\ n\in\mathbb{N}. \tag{2.2}$$

Remark 2.2. If $\{X_{nk}, n\in\mathbb{N}, k=1,\ldots,k_n\}$ is w.m.d. by the r.v. X, then

$$\frac{1}{k_n}\sum_{k=1}^{k_n}\boldsymbol{P}(\|X_{nk}\|\geqslant t)\leqslant\gamma\boldsymbol{P}(|X|\geqslant t)\quad\text{for all}\ t>0\ \text{and}\ n\in\mathbb{N}. \tag{2.3}$$

3. A general convergence rate theorem

Our main result is Theorem 3.1. It concerns the case of $k_n\equiv n$. However, its general setup allows us to apply it for general sequences k_n (see Corollaries 3.2, 3.3, 5.2 and Theorem 5.5).

Theorem 3.1. *Let $\{X_{nk}, n\in\mathbb{N}, k=1,\ldots,n\}$ be an array of rowwise independent B-valued r.v.'s which is w.m.d. by the r.v. X. Assume that there exists a sequence $\{\gamma_n, n\in\mathbb{N}\}$ of positive real numbers such that $\{\|S_n\|/\gamma_n, n\in\mathbb{N}\}$ is bounded in probability. Let $\alpha,\vartheta,\varphi\in\Phi_0$, and assume that α is not bounded, $\vartheta,\varphi\sim\Delta_2$, $\vartheta\not\equiv 0$. Let*

$$\beta(n)=\varphi(\alpha(n+1))-\varphi(\alpha(n)),\quad n=0,1,2,\ldots.$$

Assume that

$$\boldsymbol{E}\,\varphi(|X|)<\infty,\quad \boldsymbol{E}\,\vartheta(|X|)<\infty\quad\text{and}\quad\lim_{n\to\infty}\frac{\alpha(n)}{\gamma_n}=\infty.$$

Let either

$$\mu(n)=\beta(n-1)\quad\text{for all}\quad n\in\mathbb{N} \tag{3.1}$$

or

$$\mu(n)=\beta(n)\quad\text{for all}\quad n\in\mathbb{N}. \tag{3.2}$$

In case (3.2) *assume that there exists a constant $c>0$ such that for $n\in\mathbb{N}$ large enough*

$$c\beta(n)\leqslant\beta(n-1). \tag{3.3}$$

Let $n_0\in\mathbb{N}$ be such that $\vartheta(\alpha(n))>0$ for all $n\geqslant n_0$. If there exist $j\in\mathbb{N}$ and $r>0$ such that

$$\sum_{n=n_0}^{\infty}\frac{\mu(n)}{n}\left(\frac{rn+\vartheta(\gamma_n)}{\vartheta(\alpha(n))}\right)^{2^j}<\infty, \tag{3.4}$$

then

$$\sum_{n=1}^{\infty} \frac{\mu(n)}{n} \boldsymbol{P}(\|S_n\| > \varepsilon\alpha(n)) < \infty \quad \text{for all} \quad \varepsilon > 0. \tag{3.5}$$

The following corollary is a generalization of Theorem 6.2 of Fazekas (1992).

Corollary 3.2. *Let $\{X_{nk}, n \in \mathbb{N}, k = 1, \ldots, k_n\}$ be an array of rowwise independent B-valued r.v.'s which is w.m.d. by the r.v. X. Let $M \circ \psi \sim \Delta_2$, $r, s, t > 0$, $rs > t$. Assume that $\{\|S_{k_n}\|/k_n^{1/s}, n \in \mathbb{N}\}$ is bounded in probability. Furthermore, if $r > 2$ we assume that $\{M(n)/M(n-1), n \in \mathbb{N}\}$ is bounded. If*

$$\boldsymbol{E} M^{r/2}(\psi(|X|^{t/r})) < \infty \quad \text{and} \quad \boldsymbol{E}|X|^s < \infty,$$

then

$$\sum_{n=1}^{\infty} (M(n))^{r/2-1} \boldsymbol{P}(\|S_{k_n}\| > \varepsilon k_n^{r/t}) < \infty \quad \text{for all} \quad \varepsilon > 0.$$

The following corollary is a version of Corollary 4.1 of Hu et al. (1999).

Corollary 3.3. *Let $\{X_{nk}, n \in \mathbb{N}, k = 1, \ldots, k_n\}$ be an array of rowwise independent B-valued r.v.'s which is w.m.d. by the r.v. X. Let $r \in \mathbb{R}$, $0 < t < s$ and $M_r \circ \psi \sim \Delta_2$. Assume that $\{\|S_{k_n}\|/k_n^{1/s}, n \in \mathbb{N}\}$ is bounded in probability. If*

$$\boldsymbol{E} M_r(\psi(|X|^t)) < \infty \quad \text{and} \quad \boldsymbol{E}|X|^s < \infty,$$

then

$$\sum_{n=1}^{\infty} k_n^{r-2} \boldsymbol{P}(\|S_{k_n}\| > \varepsilon k_n^{1/t}) < \infty \quad \text{for all} \quad \varepsilon > 0.$$

4. Proofs

We start with some preliminary results. The following lemma is a version of Lemma 2.2 of Jain (1975).

Lemma 4.1. *Let X be a r.v., $\varphi, \alpha \in \Phi_0$, $\beta(n) = \varphi(\alpha(n+1)) - \varphi(\alpha(n))$, $n = 0, 1, 2, \ldots$. If $\boldsymbol{E}\,\varphi(|X|) < \infty$, then*

$$\sum_{n=1}^{\infty} \beta(n-1) \boldsymbol{P}(|X| \geqslant \alpha(n)) < \infty.$$

Proof. With notation $\Theta_n = \varphi(\alpha(n))$ we have

$$\begin{aligned} \boldsymbol{E}\,\varphi(|X|) &\geqslant \sum_{i=1}^{\infty} \Theta_i \boldsymbol{P}(\Theta_i \leqslant \varphi(|X|) < \Theta_{i+1}) \geqslant \sum_{i=1}^{\infty} \sum_{n=1}^{i} \beta(n-1) \boldsymbol{P}(\Theta_i \leqslant \varphi(|X|) < \Theta_{i+1}) \\ &= \sum_{n=1}^{\infty} \beta(n-1) \sum_{i=n}^{\infty} \boldsymbol{P}(\Theta_i \leqslant \varphi(|X|) < \Theta_{i+1}) \geqslant \sum_{n=1}^{\infty} \beta(n-1) \boldsymbol{P}(|X| \geqslant \alpha(n)). \qquad \square \end{aligned}$$

The following lemma is due to Hoffmann–Jørgensen (1974) and Jain (1975).

Lemma 4.2. *Let $X_1, \ldots, X_n$ be B-valued, independent, symmetric r.v.'s and $j \in \mathbb{N}$. Then there exists $A_j, B_j \geqslant 0$, depending only on j, such that*

$$\boldsymbol{P}\left(\left\|\sum_{k=1}^{n} X_k\right\| > 3^j t\right) \leqslant A_j \boldsymbol{P}\left(\max_{1\leqslant k\leqslant n} \|X_k\| > t\right) + B_j \boldsymbol{P}^{2^j}\left(\left\|\sum_{k=1}^{n} X_k\right\| > t\right)$$

for all $t \geqslant 0$. ($A_1 = 1$, $B_1 = 4$)

The following lemma is a generalization of Theorem 3.1 of Jain (1975) and Lemma 2.6 of Fazekas (1992).

Lemma 4.3. *Let $\{X_{nk}, n \in \mathbb{N}, k = 1, \ldots, k_n\}$ be an array of rowwise independent, symmetric B-valued r.v.'s and let $\{\gamma_n, n \in \mathbb{N}\}$ be a sequence of positive real numbers. Let $\vartheta \in \Phi_0$ and $\vartheta \sim \Delta_2$. If $\{\|S_{k_n}\|/\gamma_{k_n}, n \in \mathbb{N}\}$ is bounded in probability, then there exist constants $a, b > 0$ such that*

$$\boldsymbol{E}\,\vartheta(\|S_{k_n}\|) \leqslant a\boldsymbol{E}\,\vartheta\left(\max_{1\leqslant k\leqslant k_n} \|X_{nk}\|\right) + b\vartheta(\gamma_{k_n}) \quad \textit{for all} \quad n \in \mathbb{N}.$$

Proof. Let $N_{k_n} = \max_{1\leqslant k\leqslant k_n} \|X_{nk}\|$. By $\vartheta \in \Phi_0$ and Lemma 4.2, we have for all $x \geqslant 0$ and $n \in \mathbb{N}$

$$\begin{aligned}\boldsymbol{P}(\vartheta(\|S_{k_n}\|/3) > \vartheta(x)) &\leqslant \boldsymbol{P}(\|S_{k_n}\|/3 > x) \leqslant \boldsymbol{P}(N_{k_n} > x) + 4\boldsymbol{P}^2(\|S_{k_n}\| > x)\\ &\leqslant \boldsymbol{P}(\vartheta(N_{k_n}) \geqslant \vartheta(x)) + 4\boldsymbol{P}^2(\vartheta(\|S_{k_n}\|) \geqslant \vartheta(x)).\end{aligned}$$

Hence

$$\boldsymbol{P}(\vartheta(\|S_{k_n}\|/3) > t) \leqslant \boldsymbol{P}(\vartheta(N_{k_n}) \geqslant t) + 4\boldsymbol{P}^2(\vartheta(\|S_{k_n}\|) \geqslant t) \tag{4.1}$$

for all $t \in R_\vartheta$ and $n \in \mathbb{N}$. Now we prove (4.1) for $t \notin R_\vartheta$. First assume that $t \in (\vartheta(0), \sup R_\vartheta) \cap \overline{R_\vartheta}$. Then there exists $a \geqslant 0$, so that $\lim_{x\to a-0} \vartheta(x) < t < \lim_{x\to a+0} \vartheta(x)$. (Define $\lim_{x\to 0-0} \vartheta(x)$ as $\vartheta(0)$.) If $\vartheta(a) < t$, then $\bigcup_{m=1}^{\infty} \{y : \vartheta(y) > \vartheta(a + 1/m)\} = \{y : \vartheta(y) > t\}$ and $\bigcup_{m=1}^{\infty} \{y : \vartheta(y) \geqslant \vartheta(a + 1/m)\} = \{y : \vartheta(y) \geqslant t\}$. On the other hand, if $\vartheta(a) > t$, then $\bigcap_{m=1}^{\infty} \{y : \vartheta(y) > \vartheta(a - 1/m)\} = \{y : \vartheta(y) > t\}$ and $\bigcap_{m=1}^{\infty} \{y : \vartheta(y) \geqslant \vartheta(a - 1/m)\} = \{y : \vartheta(y) \geqslant t\}$. Hence, using continuity of probability and (4.1) for $t \in R_\vartheta$, we have that (4.1) is true in this case as well. If $0 \leqslant t \leqslant \vartheta(0)$ or $t \geqslant \sup R_\vartheta$, then (4.1) is obvious. Now, applying $\vartheta \sim \Delta_2$, we get that there exists a constant $c > 1$ such that

$$\boldsymbol{P}\big(\vartheta(\|S_{k_n}\|) > ct\big) \leqslant \boldsymbol{P}\big(\vartheta(N_{k_n}) \geqslant t\big) + 4\boldsymbol{P}^2\big(\vartheta(\|S_{k_n}\|) \geqslant t\big) \tag{4.2}$$

for all $t \geqslant 0$. Integrating with respect to t, we obtain

$$\frac{1}{c}\boldsymbol{E}\,\vartheta(\|S_{k_n}\|) \leqslant \boldsymbol{E}\,\vartheta(N_{k_n}) + 4\int_0^\infty \boldsymbol{P}^2\big(\vartheta(\|S_{k_n}\|) > t\big)\,\mathrm{d}t. \tag{4.3}$$

Since $\{\|S_{k_n}\|/\gamma_{k_n}, n \in \mathbb{N}\}$ is bounded in probability and $\vartheta \sim \Delta_2$, therefore there exist constants $A_1, A > 0$ such that

$$\boldsymbol{P}(\|S_{k_n}\| \geqslant A_1\gamma_{k_n}) < \frac{1}{8c} \quad \text{and} \quad \vartheta(A_1\gamma_{k_n}) \leqslant A\vartheta(\gamma_{k_n})$$

for all $n \in \mathbb{N}$. Hence we have

$$\boldsymbol{P}(\vartheta(\|S_{k_n}\|) > A\vartheta(\gamma_{k_n})) < \frac{1}{8c}.$$

It follows that

$$\int_0^\infty \mathbf{P}^2(\vartheta(\|S_{k_n}\|)>t)\mathrm{d}t \leqslant \int_0^{A\vartheta(\gamma_{k_n})} 1\mathrm{d}t + \int_{A\vartheta(\gamma_{k_n})}^\infty \frac{1}{8c}\mathbf{P}(\vartheta(\|S_{k_n}\|)>t)\mathrm{d}t$$

$$\leqslant A\vartheta(\gamma_{k_n}) + \frac{1}{8c}\mathbf{E}\,\vartheta(\|S_{k_n}\|). \tag{4.4}$$

Thus, by (4.3) and (4.4), we get Lemma 4.3. □

The following lemma is a generalization of Lemma 2.1 of Gut (1992) and Lemma 2.7 (b) of Fazekas (1992).

Lemma 4.4. *Let $\{X_{nk}, n\in\mathbb{N}, k=1,\ldots,k_n\}$ be an array of B-valued r.v.'s which is w.m.d. by the r.v. X. If $\vartheta\in\Phi_0$ then*

$$\frac{1}{k_n}\sum_{k=1}^{k_n}\mathbf{E}\,\vartheta(\|X_{nk}\|)\leqslant(1\vee\gamma)\mathbf{E}\,\vartheta(|X|). \tag{4.5}$$

Proof. Using $\vartheta\in\Phi_0$ and (2.2), we have for all $x\geqslant 0$

$$\frac{1}{k_n}\sum_{k=1}^{k_n}\mathbf{P}(\vartheta(\|X_{nk}\|)>\vartheta(x))\leqslant\frac{1}{k_n}\sum_{k=1}^{k_n}\mathbf{P}(\|X_{nk}\|>x)\leqslant\gamma\mathbf{P}(|X|>x)\leqslant\gamma\mathbf{P}(\vartheta(|X|)\geqslant\vartheta(x)),$$

hence we obtain (4.5) for $t\in R_\vartheta$. A standard calculation gives (4.5) for $t\notin R_\vartheta$. □

In the proof of Theorem 3.1 we shall apply Lemmas 4.3 and 4.4 for $k_n\equiv n$.

Proof of Theorem 3.1. First assume that X_{nk} are symmetric. Let $\varepsilon>0$. Using Lemma 4.2 and (2.2), we get

$$\mathbf{P}(\|S_n\|>\varepsilon 3^j\alpha(n))\leqslant A_j\gamma n\mathbf{P}(|X|>\varepsilon\alpha(n)) + B_j\mathbf{P}^{2^j}(\|S_n\|>\varepsilon\alpha(n)). \tag{4.6}$$

To estimate the second term of (4.6) we can apply $\vartheta\in\Phi_0$, $\vartheta\sim\Delta_2$, Chebyshev's inequality, Lemmas 4.3 and 4.4. Thus there exist $\varepsilon',\gamma',a,b>0$ such that for all $n\geqslant n_0$

$$\mathbf{P}\left(\frac{1}{\varepsilon}\|S_n\|>\alpha(n)\right)\leqslant\mathbf{P}(\varepsilon'\vartheta(\|S_n\|)\geqslant\vartheta(\alpha(n)))$$

$$\leqslant\varepsilon'\frac{\mathbf{E}\,\vartheta(\|S_n\|)}{\vartheta(\alpha(n))}\leqslant\frac{\varepsilon'}{\vartheta(\alpha(n))}(a\gamma' n\mathbf{E}\,\vartheta(|X|)+b\vartheta(\gamma_n)). \tag{4.7}$$

In formula (4.7) we can choose b such that

$$b>\frac{a}{r}\gamma'\mathbf{E}\,\vartheta(|X|), \tag{4.8}$$

where r is from (3.4). Now (4.6)–(4.8) imply that

$$\sum_{n=1}^{\infty} \frac{\mu(n)}{n} \boldsymbol{P}(\|S_n\| > \varepsilon 3^j \alpha(n)) \leqslant A_j \gamma \sum_{n=1}^{\infty} \mu(n) \boldsymbol{P}(|X| > \varepsilon\alpha(n)) + \text{const.}$$
$$+ \text{const.} \sum_{n=n_0+1}^{\infty} \frac{\mu(n)}{n} \left(\frac{rn + \vartheta(\gamma_n)}{\vartheta(\alpha(n))} \right)^{2^j}. \tag{4.9}$$

Since $\varphi \sim \Delta_2$, there exists $k>0$ such that $\boldsymbol{E}\,\varphi\big(|X|/\varepsilon\big) \leqslant k\boldsymbol{E}\,\varphi(|X|) < \infty$. Thus, by Lemma 4.1 and (3.3), there exists $n_1 \in \mathbb{N}$ such that

$$\infty > \sum_{n=1}^{\infty} \beta(n-1) \boldsymbol{P}\left(\frac{|X|}{\varepsilon} > \alpha(n) \right) \geqslant \text{const.} \sum_{n=n_1}^{\infty} \mu(n) \boldsymbol{P}(|X| > \varepsilon\alpha(n)). \tag{4.10}$$

Then (4.9), (4.10) and (3.4) imply (3.5).

In the general case let X'_{nk} be an independent copy of X_{nk} for any $n \in \mathbb{N}$ and $k = 1, \ldots, n$. Let $X^*_{nk} = X_{nk} - X'_{nk}$, $S'_n = \sum_{k=1}^n X'_{nk}$ and $S^*_n = \sum_{k=1}^n X^*_{nk} = S_n - S'_n$.

Now prove that conditions of Theorem 3.1 hold for X^*_{nk}. Using (2.1) and (2.2), we get

$$\frac{1}{n} \sum_{k=1}^{n} \boldsymbol{P}\big(\big\|X^*_{nk}\big\| > t\big) \leqslant \frac{2}{n} \sum_{k=1}^{n} \boldsymbol{P}\left(\|X_{nk}\| > \frac{t}{2} \right) \leqslant 2\gamma \boldsymbol{P}(|2X| > t) \quad \text{for all} \quad t \geqslant 0,$$

so $\{X^*_{nk} : n \in \mathbb{N}, k = 1, \ldots, n\}$ is w.m.d. by $2X$. Moreover, it follows from $\varphi, \vartheta \sim \Delta_2$ that $\boldsymbol{E}\,\varphi(|2X|) < \infty$ and $\boldsymbol{E}\,\vartheta(|2X|) < \infty$.

Since $\{\|S_n\|/\gamma_n, n \in \mathbb{N}\}$ is bounded in probability, using (2.1), for every $h>0$ there exists $q>0$ such that for all $n \in \mathbb{N}$

$$2h > 2\boldsymbol{P}(\|S_n\| > q\gamma_n) \geqslant \boldsymbol{P}(\big\|S^*_n\big\| > 2q\gamma_n).$$

Thus $\big\{\big\|S^*_n\big\|/\gamma_n, n \in \mathbb{N}\big\}$ is bounded in probability. Therefore the already known symmetric case implies

$$\sum_{n=1}^{\infty} \frac{\mu(n)}{n} \boldsymbol{P}(\big\|S^*_n\big\| > \varepsilon\alpha(n)) < \infty \quad \text{for all} \quad \varepsilon > 0. \tag{4.11}$$

Now we turn to S_n. $\big\{\big\|S'_n\big\|/\gamma_n, n \in \mathbb{N}\big\}$ is bounded in probability as well, so there exists $q' > 0$ such that

$$\boldsymbol{P}(\big\|S'_n\big\| < q'\gamma_n) > \tfrac{1}{2}. \tag{4.12}$$

Finally, (2.1), $\alpha(n)/\gamma_n \to \infty$ and (4.12) imply that for $n \in \mathbb{N}$ large enough

$$\boldsymbol{P}(\big\|S^*_n\big\| > \varepsilon\alpha(n)) \geqslant \boldsymbol{P}(\big\|S'_n\big\| < \varepsilon\alpha(n)) \boldsymbol{P}(\|S_n\| > 2\varepsilon\alpha(n))$$
$$\geqslant \boldsymbol{P}(\big\|S'_n\big\| < q'\gamma_n) \boldsymbol{P}(\|S_n\| > 2\varepsilon\alpha(n)) \geqslant \tfrac{1}{2} \boldsymbol{P}(\|S_n\| > 2\varepsilon\alpha(n)).$$

This fact and (4.11) imply (3.5). □

Proof of Corollary 3.2. In Theorem 3.1 put $\alpha(x) = x^{r/t}$, $\varphi(x) = M^{r/2}(\psi(x^{t/r}))$, $\vartheta(x) = x^s$ and $\gamma_n = n^{1/s}$. Then

$$\beta(k_n - 1) = \varphi(\alpha(k_n)) - \varphi(\alpha(k_n - 1)) = M^{r/2}(n) - M^{r/2}(n-1) \tag{4.13}$$

and $\beta(m-1) = 0$ if $k_n < m < k_{n+1}$ for all $n \in \mathbb{N}$. Using relation $M^v(n) - M^v(n-1) =$

$\int_{M(n-1)}^{M(n)} vt^{v-1}\mathrm{d}t$, it is easy to see that

$$vk_nM^{v-1}(n-1)\leqslant M^v(n)-M^v(n-1)\leqslant vk_n^{2v-1}\quad \text{for all}\quad v\geqslant 1,\ n\in\mathbb{N}, \tag{4.14}$$

and

$$vk_nM^{v-1}(n)\leqslant M^v(n)-M^v(n-1)\leqslant vk_n\quad \text{for all}\quad 0<v\leqslant 1,\ n\in\mathbb{N}. \tag{4.15}$$

Let $j\in\mathbb{N}$ be such that $2^j>t((r-1)\vee 1)/(rs-t)$. Then, using (4.13)–(4.15), we have

$$\sum_{n=1}^{\infty}\frac{\beta(k_n-1)}{k_n}\left(\frac{rk_n+\vartheta(\gamma_{k_n})}{\vartheta(\alpha(k_n))}\right)^{2^j}\leqslant\begin{cases}\text{const.}\sum_{n=1}^{\infty} n^{r-2-(sr/t-1)2^j}<\infty, & \text{if } r>2,\\ \text{const.}\sum_{n=1}^{\infty} n^{-(sr/t-1)2^j}<\infty, & \text{if } 0<r\leqslant 2.\end{cases}$$

It is easy to see that the other conditions of Theorem 3.1 are satisfied as well. Thus

$$\sum_{n=1}^{\infty}\frac{\beta(k_n-1)}{k_n}\boldsymbol{P}(\|S_{k_n}\|>\varepsilon k_n^{r/t})<\infty\quad\text{for all}\quad \varepsilon>0.$$

Furthermore, by (4.13)–(4.15), we have $\beta(k_n-1)/k_n\geqslant\text{const.}\,(M(n))^{r/2-1}$ which implies the statement. □

Proof of Corollary 3.3. In Theorem 3.1 put $\alpha(x)=x^{1/t}$, $\varphi(x)=M_r(\psi(x^t))$, $\vartheta(x)=x^s$ and $\gamma_n=n^{1/s}$. It is easy to see that the conditions of Theorem 3.1 are satisfied. Thus Theorem 3.1 implies the statement, because in this case $\beta(k_n-1)/k_n=k_n^{r-2}$ and $\beta(m-1)=0$ if $k_n<m<k_{n+1}$. □

5. Special cases of the main theorem

B is said to be of (*Rademacher*) *type p* $(0<p\leqslant 2)$ if there exists a $c>0$ such that

$$\boldsymbol{E}\left\|\sum_{i=1}^{n}X_i\right\|^p\leqslant c\sum_{i=1}^{n}\boldsymbol{E}\|X_i\|^p \tag{5.1}$$

for every independent B-valued r.v.'s $X_1,\dots,X_n$ with $\boldsymbol{E}\|X_i\|^p<\infty$ (and $\boldsymbol{E}X_i=\mathbf{0}$ if $p\geqslant 1$), $i=1,\dots,n$.

The following remark shows that in Theorem 3.1 we can write moment conditions instead of the boundedness of $\{\|S_{k_n}\|/\gamma_{k_n}, n\in\mathbb{N}\}$ if B is of type p.

Remark 5.1. *Let B be of type p for some $0<p\leqslant 2$. Let $\{X_{nk}, n\in\mathbb{N}, k=1,\dots,k_n\}$ be an array of rowwise independent B-valued* r.v.'s *which is* w.m.d. *by the* r.v. *X. Assume that $\boldsymbol{E}X_{nk}=\mathbf{0}$ $(k=1,\dots,k_n)$ when $p\geqslant 1$. If $\boldsymbol{E}|X|^p<\infty$ then $\{\|S_{k_n}\|/k_n^{1/p}, n\in\mathbb{N}\}$ is bounded in probability.*

Proof. Using (5.1) and Lemma 4.4, we have

$$\boldsymbol{E}\|S_{k_n}\|^p\leqslant c\sum_{k=1}^{k_n}\boldsymbol{E}\|X_{nk}\|^p\leqslant c(1\vee\gamma)k_n\boldsymbol{E}|X|^p.$$

So $\{\|S_{k_n}\|^p/k_n, n\in\mathbb{N}\}$ is bounded in probability. □

The following corollary is a version of Corollary 4.2 of Hu et al. (1999).

Corollary 5.2. *Let B be of type p for some $0<p\leqslant 2$. Let $\{X_{nk}, n\in\mathbb{N}, k=1,\dots,k_n\}$ be an array of rowwise independent B-valued r.v.'s which is w.m.d. by the r.v. X. Let $r\in\mathbb{R}$, $0<t<p$ and $M_r\circ\psi\sim\Delta_2$. If $\boldsymbol{E}X_{nk}=\mathbf{0}$ for all $n\in\mathbb{N}$, $k=1,\dots,k_n$, $\boldsymbol{E}M_r(\psi(|X|^t))<\infty$ and $\boldsymbol{E}|X|^p<\infty$, then*

$$\sum_{n=1}^{\infty} k_n^{r-2}\boldsymbol{P}(\|S_{k_n}\|>\varepsilon k_n^{1/t})<\infty \quad \text{for all} \quad \varepsilon>0.$$

Proof. It follows from Remark 5.1 that $\{\|S_{k_n}\|/k_n^{1/p}, n\in\mathbb{N}\}$ is bounded in probability. Hence conditions of Corollary 3.3 are satisfied. $\square$

The following three theorems are due to Fazekas (1992). We shall prove that they are special cases of Theorem 3.1.

Theorem 5.3 (*Fazekas, 1992, Theorem 3.1*). *Let $0<p\leqslant 2$, $s\geqslant p$, $rp>s$ and let B be of type p. Let $\{X_{nk}, n\in\mathbb{N}, k=1,\dots,n\}$ be an array of rowwise independent B-valued r.v.'s which is w.m.d. by the r.v. X. Assume that $\boldsymbol{E}X_{nk}=\mathbf{0}$ $(k=1,\dots,n)$ when $p\geqslant 1$. If $\boldsymbol{E}|X|^s<\infty$, then*

$$\sum_{n=1}^{\infty} n^{r-2}\boldsymbol{P}(\|S_n\|>\varepsilon n^{r/s})<\infty \quad \text{for all} \quad \varepsilon>0.$$

Proof. In Theorem 3.1 put $\alpha(x)=x^{r/s}$, $\varphi(x)=\vartheta(x)=x^s$ and $\gamma_n=n^{1/p}$. Let $j\in\mathbb{N}$ such that $2^j>rp/(rp-s)$. By Remark 5.1, $\{\|S_n\|/n^{1/p}, n\in\mathbb{N}\}$ is bounded in probability. It is easy to see that the other conditions of Theorem 3.1 hold true as well. $\square$

Theorem 5.4 (*Fazekas 1992, Theorem 3.5 and Jain 1975, Theorem 3.3*). *Let $\{X_{nk}, n\in\mathbb{N}, k=1,\dots,n\}$ be an array of rowwise independent B-valued r.v.'s which is w.m.d. by the r.v. X. Let $\alpha,\varphi\in\Phi_0$, which are strictly increasing, $R_\alpha=R_\varphi=[0,\infty)$ and $\varphi\sim\Delta_2$. Let $\beta(n)=\varphi(\alpha(n+1))-\varphi(\alpha(n))$ such that for some $c_1,c_2>0$*

$$c_1\leqslant c_2\beta(n+1)\leqslant\beta(n) \quad \text{for all} \quad n\in\mathbb{N}.$$

Let $\boldsymbol{E}\,\varphi(|X|)<\infty$. Assume that there exists a sequence $\{\gamma_n, n\in\mathbb{N}\}$ of positive real numbers such that $\{\|S_n\|/\gamma_n, n\in\mathbb{N}\}$ is bounded in probability, moreover there exists $\delta>0$ such that

$$\frac{n\vee\varphi(\gamma_n)}{\varphi(\alpha(n))}=\mathrm{O}((\log n)^{-\delta}\wedge(\beta(n))^{-\delta}). \tag{5.2}$$

Then

$$\sum_{n=1}^{\infty}\frac{\beta(n)}{n}\boldsymbol{P}(\|S_n\|>\varepsilon\alpha(n))<\infty \quad \text{for all} \quad \varepsilon>0. \tag{5.3}$$

Proof. In Theorem 3.1 put $\vartheta=\varphi$ and choose $j\in\mathbb{N}$ such that $2^j>2/\delta$. Then, using (5.2) and $1/\beta(n)\leqslant 1/c_1$, we get for some $m_0\in\mathbb{N}$ that

$$\sum_{n=1}^{\infty}\frac{\beta(n)}{n}\left(\frac{rn+\vartheta(\gamma_n)}{\vartheta(\alpha(n))}\right)^{2^j}\leqslant \text{const.}+\text{const.}\sum_{n=m_0}^{\infty}\frac{\beta(n)}{n}\left(\frac{r+1}{(\beta(n)\log n)^{\delta/2}}\right)^{2^j}$$
$$\leqslant \text{const.}+\text{const.}\sum_{n=m_0}^{\infty}n^{-1}(\log n)^{-\delta 2^{j-1}}<\infty.$$

It follows from (5.2) that $\text{const.}(\log n)^{\delta}\leqslant\varphi(\alpha(n))/\varphi(\gamma_n)$ for $n\in\mathbb{N}$ large enough, hence $\varphi(\alpha(n))/\varphi(\gamma_n)\to\infty$. This fact and $\varphi\sim\Delta_2$ imply that $\alpha(n)/\gamma_n\to\infty$. Consequently, Theorem 3.1 implies (5.3). □

Theorem 5.5 (Fazekas 1992, Theorem 6.2). *Let $\{X_{nk}, n\in\mathbb{N}, k=1,\dots,k_n\}$ be an array of rowwise independent B-valued r.v.'s which is w.m.d. by the r.v. X. Let $0<p\leqslant 2$, $r\geqslant 1$, $t>0$ and $s\geqslant p$. Suppose that $r>t/p$* if $s>1$ *while $r>t/s$* if $s\leqslant 1$. *Assume that*

$$\limsup_{n\to\infty}\frac{k_n}{M(n-1)}<\infty\quad \textit{if}\quad r>2.$$

Let $M\circ\psi\sim\Delta_2$ and B be of type p. Assume that $\boldsymbol{E}X_{nk}=\boldsymbol{0}$ $(k=1,\dots,k_n)$ in case $p\geqslant 1$. If

$$\boldsymbol{E}M^{r/2}(\psi(|X|^{t/r}))<\infty\quad \textit{and}\quad \boldsymbol{E}|X|^s<\infty,$$

then

$$\sum_{n=1}^{\infty}(M(n))^{r/2-1}\boldsymbol{P}(\|S_{k_n}\|>\varepsilon k_n^{r/t})<\infty\quad \textit{for all}\quad \varepsilon>0.$$

Proof. Let $q=p$ if $s>1$ while $q=s$ if $s\leqslant 1$. Then $rq>t$, B is of type q and $\boldsymbol{E}|X|^q<\infty$. Hence, using Remark 5.1, we get that $\{\|S_{k_n}\|/k_n^{1/q}, n\in\mathbb{N}\}$ is bounded in probability. On the other hand $\limsup_{n\to\infty}k_n/M(n-1)<\infty$ implies the boundedness of $\{M(n)/M(n-1), n\in\mathbb{N}\}$. So all conditions of Corollary 3.2 are satisfied. □

Remark 5.6. Theorem 5.5 can be applied e.g., for $k_n=d^n$ and $k_n=n^d$, where d is a fixed positive integer.

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References

Ahmed, S.E., Antonini, R.G., Volodin, A., 2002. On the rate of complete convergence for weighted sums of arrays of Banach space valued random elements with application to moving average processes. Statist. Probab. Lett. 58, 185–194.
Baum, L.E., Katz, M., 1965. Convergence rates in the law of large numbers. Trans. Amer. Math. Soc. 120, 108–123.
Csörgő, S., 2003. Rates in the complete convergence of bootstrap means. Statist. Probab. Lett. 64, 359–368.
Erdős, P., 1949. On a theorem of Hsu and Robbins. Ann. Math. Statist. 20, 286–291.
Erdős, P., 1950. Remark on my paper "On a theorem of Hsu and Robbins". Ann. Math. Statist. 21, 138.
Fazekas, I., 1992. Convergence rates in the law of large numbers for arrays. Publ. Math. Debrecen 41/1–2, 53–71.
Gut, A., 1985. On complete convergence in the law of large numbers for subsequences. Ann. Probab. 13, 1286–1291.

Gut, A., 1992. Complete convergence for arrays. Periodica Math. Hungar. 25 (1), 51–75.
Hoffmann-Jørgensen, J., 1974. Sums of independent Banach space valued random variables. Studia Math. LII, 159–186.
Hsu, P.L., Robbins, H., 1947. Complete convergence and the law of large numbers. Proc. Nat. Acad. Sci. USA 33, 25–31.
Hu, T.-C., Móricz, F., Taylor, R.L., 1989. Strong laws of large numbers for arrays of rowwise independent random variables. Acta Math. Acad. Sci. Hungar. 54, 153–162.
Hu, T.-C., Rosalsky, A., Szynal, D., Volodin, A.I., 1999. On complete convergence for arrays of rowwise independent random elements in Banach spaces. Stoch. Anal. Appl. 17 (6), 963–992.
Jain, N.C., 1975. Tail probabilities for sums of independent Banach space valued random variables. Z. Wahrscheinlichkeitstheorie verw. Gebiete 33, 155–166.
Spitzer, F.L., 1956. A combinatorial lemma and its application. Trans. Amer. Math. Soc. 82, 323–339.
Woyczyński, W.A., 1980. On Marcinkiewicz-Zygmund laws of large numbers in Banach spaces and related rates of convergence. Probab. Math. Statist. 1, 117–131.