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István Fazekas and Tibor Tómács

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DEPARTMENT OF MATHEMATICS

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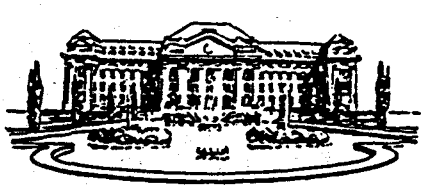

Strong Law of Large Numbers for Pairwise Independent Random Variables with Multidimensional Indices

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STRONG LAW OF LARGE NUMBERS FOR PAIRWISE INDEPENDENT RANDOM VARIABLES WITH MULTIDIMENSIONAL INDICES

ISTVÁN FAZEKAS (Debrecen) and TIBOR TÓMÁCS (Eger)

1. Introduction

Several papers are devoted to the study of the strong law of large numbers for non independent random variables (see e.g. Révész [11] and Csörgő, Tandori and Totik [1]). Etemadi [2] proved that the Kolmogorov strong law of large numbers holds for identically distributed and pairwise independent random variables. Kruglov [9] extended that result and obtained the Marcinkiewicz strong law of large numbers and Spitzer's theorem in the pairwise independent case if $r < 1$.

On the other hand, the strong law of large numbers has been extended to the case where the index set is the positive integer d-dimensional lattice points (Gut [5], Klesov [8], Fazekas [3]). Moreover, the assumption of identical distribution can also be weakened. Hu, Móricz and Taylor [7], Gut [6] and Fazekas [4] used domination of distributions instead of identical distribution.

In this paper the Kolmogorov and the Marcinkiewicz strong laws (if $0 < r < 1$) are proved for pairwise independent identically distributed random variables with multidimensional indices. Spitzer's theorem is obtained for pairwise independent dominated random variables with multidimensional indices. Our theorems are extensions of Theorems 1, 2 and 3 of Kruglov [9]. Some parts of our theorems have been proved in Etemadi [2].

2. Notation and preliminary lemmas

Let $\mathbf{N}^d$ be the positive integer d-dimensional lattice points, where d is a positive integer. For $\mathbf{n}, \mathbf{m} \in \mathbf{N}^d$, $\mathbf{n} \leq \mathbf{m}$ is defined coordinatewise, $(\mathbf{n}, \mathbf{m}] = \prod_{i=1}^d (n_i, m_i]$ is a d-dimensional rectangle and $|\mathbf{n}| = \prod_{i=1}^d n_i$ where $\mathbf{n} = (n_1, \ldots, n_d)$, $\mathbf{m} = (m_1, \ldots, m_d)$. $\sum_{\mathbf{n}}$ will denote the summation for all $\mathbf{n} \in \mathbf{N}^d$. $\mathbf{1} = (1, \ldots, 1) \in \mathbf{N}^d$. $I(A)$ denotes the indicator function of the set A. We shall assume that random variables (r.v.'s) $\{X_{\mathbf{n}}, \mathbf{n} \in \mathbf{N}^d\}$ are defined on the same probability space $(\Omega, \mathcal{A}, P)$. The following notation will be used: $S_{\mathbf{n}} = \sum_{\mathbf{k} \leq \mathbf{n}} X_{\mathbf{k}}$, $X_{\mathbf{n}}^+ = \max\{0, X_{\mathbf{n}}\}$, $X_{\mathbf{n}}^- = \max\{0, -X_{\mathbf{n}}\}$. Obviously, $X_{\mathbf{n}} = X_{\mathbf{n}}^+ - X_{\mathbf{n}}^-$, $|X_{\mathbf{n}}| = X_{\mathbf{n}}^+ + X_{\mathbf{n}}^-$.

The following two lemmas are proved in Gut [5] and in Fazekas [3].

LEMMA 1. *Let X be a r.v. For $r > 0$ and $m = 0, 1, 2, \ldots$ the following statements are equivalent:*
1) $\mathbf{E}\left(|X|^r \left(\log^+ |X|\right)^{d-1+m}\right) < \infty$,
2) $\sum_{\mathbf{n}} |\mathbf{n}|^{\alpha r - 1} (\log |\mathbf{n}|)^m \mathbf{P}(|X| \geq |\mathbf{n}|^\alpha \varepsilon) < \infty$, *for any $\alpha > 0$, $\varepsilon > 0$.*

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LEMMA 2. *Let $\{X_{\mathbf{n}}, \mathbf{n} \in \mathbf{N}^d\}$ be a sequence of identically distributed (i.d.) r.v.'s, $0 < r < p \leq 2$, $\varepsilon > 0$ and define $Y_{\mathbf{n}} = X_{\mathbf{n}} I\left\{|X_{\mathbf{n}}| \leq \varepsilon |\mathbf{n}|^{1/r}\right\}$. If*

$$\mathbf{E}\left(|X_{\mathbf{1}}|^r \left(\log^+ |X_{\mathbf{1}}|\right)^{d-1+m}\right) < \infty,$$

then

$$\sum_{\mathbf{n}} (\log |\mathbf{n}|)^m \, \mathbf{E}\left||\mathbf{n}|^{-1/r} Y_{\mathbf{n}}\right|^p < \infty \quad \text{for} \quad m = 0, 1, 2, \ldots.$$

The following lemma is a variant of Lemma 2.1. of Gut [6]. (See also Lemma 2.7. of Fazekas [4]).

LEMMA 3. *Let $\{X_{\mathbf{n}}, \mathbf{n} \in \mathbf{N}^d\}$ be a sequence of r.v.'s. Suppose that there exists a r.v. X such that*

$$\frac{1}{|\mathbf{n}|} \sum_{\mathbf{k} \leq \mathbf{n}} \mathbf{P}\left(|X_{\mathbf{k}}| > x\right) \leq c \mathbf{P}\left(|X| > x\right)$$

for all $\mathbf{n} \in \mathbf{N}^d$, $x > 0$ and some $c > 0$. Let $p > 0$ and define, for $\lambda > 0$, $X_{\mathbf{n}}(\lambda) = |X_{\mathbf{n}}| I\{|X_{\mathbf{n}}| > \lambda\}$, $X^*_{\mathbf{n}}(\lambda) = |X_{\mathbf{n}}| I\{|X_{\mathbf{n}}| \leq \lambda\} + \lambda I\{|X_{\mathbf{n}}| > \lambda\}$ and similarly for X. Then

$$\frac{1}{|\mathbf{n}|} \sum_{\mathbf{k} \leq \mathbf{n}} \mathbf{E}\left(X^*_{\mathbf{k}}(\lambda)\right)^p \leq c \mathbf{E}\left(X^*(\lambda)\right)^p, \tag{1}$$

and

$$\frac{1}{|\mathbf{n}|} \sum_{\mathbf{k} \leq \mathbf{n}} \mathbf{E}\left(X_{\mathbf{k}}(\lambda)\right)^p \leq c \mathbf{E}\left(X(\lambda)\right)^p. \tag{2}$$

PROOF. For a non-negative r.v. Y we have

$$\mathbf{E} Y^p = p \int_0^\infty y^{p-1} \mathbf{P}(Y > y) dy.$$

By this equality and our condition it follows that

$$\frac{1}{|\mathbf{n}|} \sum_{\mathbf{k} \leq \mathbf{n}} \mathbf{E}\left(X^*_{\mathbf{k}}(\lambda)\right)^p = \frac{1}{|\mathbf{n}|} \sum_{\mathbf{k} \leq \mathbf{n}} p \int_0^\infty y^{p-1} \mathbf{P}\left(X^*_{\mathbf{k}}(\lambda) > y\right) dy$$

$$= p \int_0^\lambda y^{p-1} \frac{1}{|\mathbf{n}|} \sum_{\mathbf{k} \leq \mathbf{n}} \mathbf{P}\left(|X_{\mathbf{k}}| > y\right) dy \leq p \int_0^\lambda y^{p-1} c \mathbf{P}(|X| > y) dy$$

$$= c \mathbf{E}\left(X^*(\lambda)\right)^p.$$

Therefore (1) is proved. Similarly, (2) follows, since

$$\frac{1}{|\mathbf{n}|} \sum_{\mathbf{k} \leq \mathbf{n}} \mathbf{E}\left(X_{\mathbf{k}}(\lambda)\right)^p = p \int_0^\infty y^{p-1} \frac{1}{|\mathbf{n}|} \sum_{\mathbf{k} \leq \mathbf{n}} \mathbf{P}\left(X_{\mathbf{k}}(\lambda) > y\right) dy$$

$$= p \int_0^\lambda y^{p-1} \frac{1}{|\mathbf{n}|} \sum_{\mathbf{k} \leq \mathbf{n}} \mathbf{P}\left(|X_{\mathbf{k}}| > \lambda\right) dy + p \int_\lambda^\infty y^{p-1} \frac{1}{|\mathbf{n}|} \sum_{\mathbf{k} \leq \mathbf{n}} \mathbf{P}\left(|X_{\mathbf{k}}| > y\right) dy$$

$$\leq p \int_0^\infty y^{p-1} c \mathbf{P}(X(\lambda) > y) dy = c \mathbf{E}(X(\lambda))^p.$$

This completes the proof of Lemma 3.

LEMMA 4. *Let $\{X_{\mathbf{n}}, \mathbf{n} \in \mathbf{N}^d\}$ be a sequence of pairwise independent r.v.'s, and let $\{a_{\mathbf{n}}, \mathbf{n} \in \mathbf{N}^d\}$ be a sequence of positive numbers. If*

$$\left\{\frac{a_{\mathbf{n}-\mathbf{v}}}{a_{\mathbf{n}}} : \mathbf{n} \in \mathbf{N}^d, \mathbf{v} \in V\right\}$$

is a bounded set, where $V = \{\mathbf{v} = (v_1, \ldots, v_d) : v_i \in \{0,1\}\}$ and

$$\frac{S_{\mathbf{n}}}{a_{\mathbf{n}}} \to 0 \quad \text{almost surely (a.s.)} \quad \text{as} \quad |\mathbf{n}| \to \infty,$$

then

$$\sum_{\mathbf{n}} \mathbf{P}\left(|X_{\mathbf{n}}| \geq a_{\mathbf{n}}\right) < \infty.$$

PROOF. This lemma has been proved in Petrov [10] (p. 222) for $d = 1$. The reader can readily verify that

$$\frac{X_{\mathbf{n}}}{a_{\mathbf{n}}} = (-1)^{\sum_{i=1}^{d} v_i} \sum_{\mathbf{v} \in V} \frac{S_{\mathbf{n}-\mathbf{v}}}{a_{\mathbf{n}-\mathbf{v}}} \frac{a_{\mathbf{n}-\mathbf{v}}}{a_{\mathbf{n}}}.$$

So $X_{\mathbf{n}}/a_{\mathbf{n}} \to 0$ a.s. as $|\mathbf{n}| \to \infty$. It implies

$$\mathbf{P}\Big(\{|X_{\mathbf{n}}| \geq a_{\mathbf{n}}\} \text{ for finitely many values of } \mathbf{n}\Big) = 1.$$

So the lemma follows from Borel–Cantelli lemma for pairwise independent events (see e.g. Petrov [10] p. 214). This completes the proof of Lemma 4.

LEMMA 5. (Kronecker) Let $x_{\mathbf{n}}$ and $b_{\mathbf{n}}$ non-negative numbers ($\mathbf{n} \in \mathbf{N}^d$). Suppose that $b_{\mathbf{m}} \leq b_{\mathbf{n}}$ if $\mathbf{m} \leq \mathbf{n}$ and $b_{\mathbf{n}} \to \infty$ if $|\mathbf{n}| \to \infty$. If $\sum_{\mathbf{n}} x_{\mathbf{n}}$ is finite then

$$\frac{1}{b_{\mathbf{n}}} \sum_{\mathbf{k} \leq \mathbf{n}} b_{\mathbf{k}} x_{\mathbf{k}} \to 0 \quad \text{as} \quad |\mathbf{n}| \to \infty.$$

PROOF. For any $\varepsilon > 0$ there exists $\mathbf{n}_\varepsilon \in \mathbf{N}^d$ such that $\sum_{\mathbf{n} \nleq \mathbf{n}_\varepsilon} x_{\mathbf{n}} < \varepsilon$. Therefore

$$0 \leq \frac{1}{b_{\mathbf{n}}} \sum_{\mathbf{k} \leq \mathbf{n}} b_{\mathbf{k}} x_{\mathbf{k}} = \frac{1}{b_{\mathbf{n}}} \sum_{\substack{\mathbf{k} \leq \mathbf{n} \\ \mathbf{k} \leq \mathbf{n}_\varepsilon}} b_{\mathbf{k}} x_{\mathbf{k}} + \frac{1}{b_{\mathbf{n}}} \sum_{\substack{\mathbf{k} \leq \mathbf{n} \\ \mathbf{k} \nleq \mathbf{n}_\varepsilon}} b_{\mathbf{k}} x_{\mathbf{k}} \leq \frac{1}{b_{\mathbf{n}}} A_\varepsilon + \varepsilon \to 0$$

as $|\mathbf{n}| \to \infty$ and $\varepsilon \downarrow 0$.

3. Results

THEOREM 1. *Let $\{X_{\mathbf{n}}, \mathbf{n} \in \mathbf{N}^d\}$ be a sequence of non-negative r.v.'s, and let $\{b_{\mathbf{n}}, \mathbf{n} \in \mathbf{N}^d\}$ be a bounded sequence of non-negative numbers, and $B_{\mathbf{n}} = \sum_{\mathbf{k} \leq \mathbf{n}} b_{\mathbf{k}}$. If*

$$\sum_{\mathbf{n}} \frac{1}{|\mathbf{n}|} \mathbf{P}\left(|S_{\mathbf{n}} - B_{\mathbf{n}}| > \varepsilon |\mathbf{n}|^{1/r}\right) < \infty \tag{3}$$

for every $\varepsilon > 0$, where $0 < r \leq 1$, then

$$\frac{1}{|\mathbf{n}|^{1/r}} (S_{\mathbf{n}} - B_{\mathbf{n}}) \to 0 \quad \text{a.s.} \quad \text{as} \quad |\mathbf{n}| \to \infty. \tag{4}$$

PROOF. Fix $\alpha > 1$, $\varepsilon > 0$, denote the integer part of α^{n_i} by k_{n_i} $(i = 1, \ldots, d)$ and $\mathbf{k_n} = (k_{n_1}, \ldots, k_{n_d})$. It follows from the inequalities

$$\begin{aligned}
&\sum_{\mathbf{n}} \frac{|\mathbf{k_{n+1}} - \mathbf{k_n}|}{|\mathbf{k_{n+1}}|} \min_{\mathbf{k} \in (\mathbf{k_n}, \mathbf{k_{n+1}}]} \mathbf{P}\left(|S_{\mathbf{k}} - B_{\mathbf{k}}| > \varepsilon |\mathbf{k}|^{1/r}\right) \\
&\leq \sum_{\mathbf{n}} \sum_{\mathbf{k} \in (\mathbf{k_n}, \mathbf{k_{n+1}}]} \frac{1}{|\mathbf{k}|} \mathbf{P}\left(|S_{\mathbf{k}} - B_{\mathbf{k}}| > \varepsilon |\mathbf{k}|^{1/r}\right) \\
&\leq \sum_{\mathbf{n}} \frac{1}{|\mathbf{n}|} \mathbf{P}\left(|S_{\mathbf{n}} - B_{\mathbf{n}}| > \varepsilon |\mathbf{n}|^{1/r}\right)
\end{aligned}$$

and condition (3) that there exists a sequence $\mathbf{m_n} = (m_{n_1}, \ldots, m_{n_d})$, $\alpha^{n_i} < m_{n_i} \leq \alpha^{n_i+1}$ $(i = 1, \ldots, d)$ such that the series

$$\sum_{\mathbf{n}} \mathbf{P}\left(|S_{\mathbf{m_n}} - B_{\mathbf{m_n}}| > \varepsilon |\mathbf{m_n}|^{1/r}\right) \tag{5}$$

converges. By the Borel–Cantelli lemma convergence of the series (5) implies

$$\frac{1}{|\mathbf{m_n}|^{1/r}} |S_{\mathbf{m_n}} - B_{\mathbf{m_n}}| \leq \varepsilon \tag{6}$$

except for finitely many values of $\mathbf{m_n}$ a.s. For any $\mathbf{t} \in \mathbf{N}^d$ there exists an index $\mathbf{n} \in \mathbf{N}^d$ such that $\mathbf{t} \in (\mathbf{m_n}, \mathbf{m_{n+1}}]$. By non-negativity of $X_{\mathbf{k}}$ we have

$$\begin{aligned}
&\frac{1}{|\mathbf{t}|^{1/r}} (B_{\mathbf{m_n}} - B_{\mathbf{t}}) + \frac{1}{|\mathbf{t}|^{1/r}} (S_{\mathbf{m_n}} - B_{\mathbf{m_n}}) \leq \frac{1}{|\mathbf{t}|^{1/r}} (S_{\mathbf{t}} - B_{\mathbf{t}}) \\
&\leq \frac{1}{|\mathbf{t}|^{1/r}} \left(S_{\mathbf{m_{n+1}}} - B_{\mathbf{m_{n+1}}}\right) + \frac{1}{|\mathbf{t}|^{1/r}} \left(B_{\mathbf{m_{n+1}}} - B_{\mathbf{t}}\right).
\end{aligned} \tag{7}$$

Put $b = \sup_{\mathbf{n}} b_{\mathbf{n}}$ and observe that

$$\frac{1}{|\mathbf{t}|^{1/r}} (B_{\mathbf{t}} - B_{\mathbf{m_n}}) \leq \left(\alpha^{2d} - 1\right) b \alpha^{(1-1/r) \sum_{i=1}^{d} n_i} \leq \left(\alpha^{2d} - 1\right) b$$

and

$$\frac{1}{|\mathbf{t}|^{1/r}}\left(B_{\mathbf{m}_n+1}-B_{\mathbf{t}}\right)\le\left(\alpha^{2d}-1\right)b,$$

as $0<r\le 1$. In view of the inequality $|\mathbf{t}|^{-1/r}\le \alpha^{2d/r}|\mathbf{m}_{n+1}|^{-1/r}$, (6) and (7) give

$$\frac{1}{|\mathbf{t}|^{1/r}}|S_{\mathbf{t}}-B_{\mathbf{t}}|\le \varepsilon\alpha^{2d/r}+\left(\alpha^{2d}-1\right)b$$

except for finitely many $\mathbf{t}$ a.s. For any $\delta>0$ we can find an $\varepsilon>0$ and an $\alpha>1$ such that $\varepsilon\alpha^{2d/r}+\left(\alpha^{2d}-1\right)b<\delta$. Therefore (4) is proved.

THEOREM 2. *Let $\{X_{\mathbf{n}},\mathbf{n}\in\mathbf{N}^d\}$ be a sequence of pairwise independent r.v.'s. Assume that*
1) $\sup_{\mathbf{n}}\mathbf{E}|X_{\mathbf{n}}|<\infty$,
2) there exists a r.v. X such that $\mathbf{E}\left(|X|\left(\log^+|X|\right)^{d-1}\right)<\infty$ and

$$\frac{1}{|\mathbf{n}|}\sum_{\mathbf{k}\le\mathbf{n}}\mathbf{P}\left(|X_{\mathbf{k}}|>x\right)\le c\mathbf{P}\left(|X|>x\right)$$

for all $\mathbf{n}\in\mathbf{N}^d$, $x>0$ and some $c>0$. Then

$$\frac{1}{|\mathbf{n}|}\left(S_{\mathbf{n}}-\mathbf{E}S_{\mathbf{n}}\right)\to 0\quad\text{a.s.}\quad\text{as}\quad|\mathbf{n}|\to\infty \tag{8}$$

and moreover for any $\varepsilon>0$

$$\sum_{\mathbf{n}}\frac{1}{|\mathbf{n}|}\mathbf{P}\left(|S_{\mathbf{n}}-\mathbf{E}S_{\mathbf{n}}|>\varepsilon|\mathbf{n}|\right)<\infty. \tag{9}$$

PROOF. Assumption 1) is not used to prove (9). First we prove (9). Put $Y_{\mathbf{k}}=X_{\mathbf{k}}I\{|X_{\mathbf{k}}|\le|\mathbf{n}|\}$, $\mathbf{k}\le\mathbf{n}$, $T_{\mathbf{n}}=\sum_{\mathbf{k}\le\mathbf{n}}Y_{\mathbf{k}}$. Remark that pairwise independence of $X_{\mathbf{k}}$ implies that of $Y_{\mathbf{k}}$. Condition 2) and (1) in Lemma 3 (with $\lambda=|\mathbf{n}|$, $p=2$) imply the inequality

$$\frac{1}{|\mathbf{n}|}\sum_{\mathbf{k}\le\mathbf{n}}\mathbf{E}Y_{\mathbf{k}}^2\le c|\mathbf{n}|^2\mathbf{P}\left(|X|>|\mathbf{n}|\right)+c\mathbf{E}\left(X^2I\{|X|\le|\mathbf{n}|\}\right).$$

Further, we have

$$\begin{aligned}\sum_{\mathbf{n}}\frac{1}{|\mathbf{n}|^3}\mathbf{D}^2T_{\mathbf{n}}&=\sum_{\mathbf{n}}\frac{1}{|\mathbf{n}|^3}\sum_{\mathbf{k}\le\mathbf{n}}\mathbf{D}^2Y_{\mathbf{k}}\le\sum_{\mathbf{n}}\frac{1}{|\mathbf{n}|^3}\sum_{\mathbf{k}\le\mathbf{n}}\mathbf{E}Y_{\mathbf{k}}^2\\&\le c\sum_{\mathbf{n}}\mathbf{P}\left(|X|>|\mathbf{n}|\right)+c\sum_{\mathbf{n}}\frac{1}{|\mathbf{n}|^2}\mathbf{E}\left(X^2I\{|X|\le|\mathbf{n}|\}\right).\end{aligned}$$

Obviously

$$\mathbf{P}\left(|S_{\mathbf{n}}-\mathbf{E}T_{\mathbf{n}}|>\varepsilon|\mathbf{n}|\right)\le\mathbf{P}\left(|T_{\mathbf{n}}-\mathbf{E}T_{\mathbf{n}}|>\varepsilon|\mathbf{n}|\right)+\mathbf{P}\left(\bigcup_{\mathbf{k}\le\mathbf{n}}\{|X_{\mathbf{k}}|>|\mathbf{n}|\}\right).$$

Therefore by the Chebyshev inequality, Lemma 1 and Lemma 2 we have

$$\begin{aligned}
&\sum_{\mathbf{n}} \frac{1}{|\mathbf{n}|} \mathbf{P}\left(|S_{\mathbf{n}} - \mathbf{E}T_{\mathbf{n}}| > \varepsilon|\mathbf{n}|\right) \\
&\le \frac{1}{\varepsilon^2} \sum_{\mathbf{n}} \frac{1}{|\mathbf{n}|^3} \mathbf{D}^2 T_{\mathbf{n}} + \sum_{\mathbf{n}} \frac{1}{|\mathbf{n}|} \sum_{\mathbf{k} \le \mathbf{n}} \mathbf{P}\left(|X_{\mathbf{k}}| > |\mathbf{n}|\right) \\
&\le c\left(1 + \frac{1}{\varepsilon^2}\right) \sum_{\mathbf{n}} \mathbf{P}\left(|X| > |\mathbf{n}|\right) + \frac{c}{\varepsilon^2} \sum_{\mathbf{n}} \frac{1}{|\mathbf{n}|^2} \mathbf{E}\left(X^2 I\{|X| \le |\mathbf{n}|\}\right) < \infty.
\end{aligned}$$

Hence (9) follows, since

$$\begin{aligned}
\frac{1}{|\mathbf{n}|} |\mathbf{E}S_{\mathbf{n}} - \mathbf{E}T_{\mathbf{n}}| &\le \frac{1}{|\mathbf{n}|} \sum_{\mathbf{k} \le \mathbf{n}} \mathbf{E}\Big(|X_{\mathbf{k}}| I\{|X_{\mathbf{k}}| > |\mathbf{n}|\}\Big) \\
&\le c\mathbf{E}\Big(|X| I\{|X| > |\mathbf{n}|\}\Big) \to 0
\end{aligned}$$

as $|\mathbf{n}| \to \infty$ by (2) in Lemma 3 (with $\lambda = |\mathbf{n}|$, $p = 1$). Now we turn to (8). It follows from the equality $|X_{\mathbf{n}}| = X_{\mathbf{n}}^{+} + X_{\mathbf{n}}^{-}$ and condition 2) that

$$\frac{1}{|\mathbf{n}|} \sum_{\mathbf{k} \le \mathbf{n}} \mathbf{P}\left(X_{\mathbf{n}}^{\pm} > x\right) \le c\mathbf{P}\left(|X| > x\right)$$

for all $\mathbf{n} \in \mathbf{N}^d$ and $x > 0$. Therefore (9) holds with $X_{\mathbf{k}}$ replaced by $X_{\mathbf{k}}^{+}$ and $X_{\mathbf{k}}^{-}$. By Theorem 1 it follows that

$$\frac{1}{|\mathbf{n}|} \sum_{\mathbf{k} \le \mathbf{n}} \left(X_{\mathbf{k}}^{\pm} - \mathbf{E}X_{\mathbf{k}}^{\pm}\right) \to 0 \quad \text{a.s.} \quad \text{as} \quad |\mathbf{n}| \to \infty.$$

Therefore (8) is proved.

COROLLARY. *Let $\{X_{\mathbf{n}}, \mathbf{n} \in \mathbf{N}^d\}$ be a sequence of pairwise independent i.d.* r.v.'s. *The following statements are equivalent:*

1) $\mathbf{E}|X_1| \left(\log^{+} |X_1|\right)^{d-1} < \infty$,

2) $|\mathbf{n}|^{-1} S_{\mathbf{n}} \to c$ *a.s., where c is some constant,*

3) for any $\varepsilon > 0$ and some $b \ge 0$

$$\sum_{\mathbf{n}} \frac{1}{|\mathbf{n}|} \mathbf{P}\left(\left|\sum_{\mathbf{k} \le \mathbf{n}} (|X_{\mathbf{k}}| - b)\right| > \varepsilon|\mathbf{n}|\right) < \infty.$$

PROOF. Theorem 2 implies 1)⇒2) and 1)⇒3) with $b = \mathbf{E}|X_1|$. Implication 2)⇒1) is a consequence of Lemma 4 and Lemma 1. Now we prove implication 3)⇒1). By Theorem 1 we have

$$\frac{1}{|\mathbf{n}|} \sum_{\mathbf{k} \le \mathbf{n}} |X_{\mathbf{k}}| \to b \quad \text{a.s.} \quad \text{as} \quad |\mathbf{n}| \to \infty.$$

So implication 3)⇒1) follows from implication 2)⇒1).

THEOREM 3. *Let $\{X_{\mathbf{n}}, \mathbf{n} \in \mathbf{N}^d\}$ be a sequence of pairwise independent i.d. r.v.'s and $0 < r < 1$. The following statements are equivalent:*

1) $\mathbf{E}|X_{\mathbf{1}}|^r \left(\log^+ |X_{\mathbf{1}}|\right)^{d-1} < \infty$,

2) $\sum_{\mathbf{n}} |X_{\mathbf{n}}|/|\mathbf{n}|^{1/r}$ *is convergent a.s.,*

3) $|\mathbf{n}|^{-1/r} S_{\mathbf{n}} \to 0$ *a.s.,*

4) for any $\varepsilon > 0$

$$\sum_{\mathbf{n}} \frac{1}{|\mathbf{n}|} \mathbf{P}\left(\sum_{\mathbf{k}\le\mathbf{n}} |X_{\mathbf{k}}| > \varepsilon |\mathbf{n}|^{1/r}\right) < \infty.$$

PROOF. First we prove implication 3)⇒1). By Lemma 4 we have

$$\sum_{\mathbf{n}} \mathbf{P}\left(|X_{\mathbf{1}}| \ge |\mathbf{n}|^{1/r}\right) < \infty.$$

So 1) is a consequence of Lemma 1. Implication 4)⇒3) follows from Theorem 1 (pairwise independence is not necessary). Now we prove 1)⇒2) (pairwise independence is not necessary). Let $Y_{\mathbf{n}} = X_{\mathbf{n}} I\{|X_{\mathbf{n}}| \le |\mathbf{n}|^{1/r}\}$. Then by Lemma 2 we have $\sum_{\mathbf{n}} \mathbf{E}\left||\mathbf{n}|^{-1/r} Y_{\mathbf{n}}\right| < \infty$. That is $\sum_{\mathbf{n}} |\mathbf{n}|^{-1/r} |Y_{\mathbf{n}}|$ is integrable, so it is finite a.s. By Lemma 1

$$\sum_{\mathbf{n}} \mathbf{P}(X_{\mathbf{n}} \ne Y_{\mathbf{n}}) < \infty$$

therefore Borel–Cantelli lemma implies 2). Implication 2)⇒3) is a consequence of the Lemma 5 (pairwise independence is not necessary). So it remains to prove implication 1)⇒4). Put $Y_{\mathbf{k}} = |X_{\mathbf{k}}| I\left\{|X_{\mathbf{k}}| \le |\mathbf{n}|^{1/r}\right\}$, $\mathbf{k} \le \mathbf{n}$, $Z_{\mathbf{n}} = \sum_{\mathbf{k}\le\mathbf{n}} Y_{\mathbf{k}}$ and $S^*_{\mathbf{n}} = \sum_{\mathbf{k}\le\mathbf{n}} |X_{\mathbf{k}}|$. We have

$$\sum_{\mathbf{n}} \frac{\mathbf{E} Y_{\mathbf{n}}^p}{|\mathbf{n}|^{p/r}} < \infty \tag{10}$$

for any $0 < r < p \le 2$ (see Lemma 2). Further we have

$$\begin{aligned}
&\sum_{\mathbf{n}} \frac{1}{|\mathbf{n}|} \mathbf{P}\left(|S^*_{\mathbf{n}} - \mathbf{E} Z_{\mathbf{n}}| > \varepsilon |\mathbf{n}|^{1/r}\right) \\
&\le \sum_{\mathbf{n}} \frac{1}{|\mathbf{n}|} \mathbf{P}\left(|Z_{\mathbf{n}} - \mathbf{E} Z_{\mathbf{n}}| > \varepsilon |\mathbf{n}|^{1/r}\right) + \sum_{\mathbf{n}} \frac{1}{|\mathbf{n}|} \mathbf{P}\left(\bigcup_{\mathbf{k}\le\mathbf{n}} \left\{|X_{\mathbf{k}}| > |\mathbf{n}|^{1/r}\right\}\right) \\
&\le \sum_{\mathbf{n}} \frac{1}{|\mathbf{n}|} \frac{\mathbf{D}^2 Z_{\mathbf{n}}}{\varepsilon^2 |\mathbf{n}|^{2/r}} + \sum_{\mathbf{n}} \mathbf{P}\left(|X_{\mathbf{1}}| > |\mathbf{n}|^{1/r}\right) \\
&\le \frac{1}{\varepsilon^2} \sum_{\mathbf{n}} \frac{\mathbf{E} Y_{\mathbf{n}}^2}{|\mathbf{n}|^{2/r}} + \sum_{\mathbf{n}} \mathbf{P}\left(|X_{\mathbf{1}}| \ge |\mathbf{n}|^{1/r}\right) < \infty.
\end{aligned}$$

Here we used Lemma 1 and (10) with $p = 2$. By (10) it follows that for any $\varepsilon > 0$ there exists an $\mathbf{m}_\varepsilon \in \mathbf{N}^d$ such that

$$\sum_{\mathbf{n}\not\le\mathbf{m}_\varepsilon} \frac{\mathbf{E} Y_{\mathbf{n}}}{|\mathbf{n}|^{1/r}} < \varepsilon.$$

If $\mathbf{m} \not\le \mathbf{m}_\varepsilon$ then

$$\varepsilon > \sum_{\mathbf{n}\in(\mathbf{m},2\mathbf{m}]} \frac{\mathbf{E} Y_{\mathbf{n}}}{|\mathbf{n}|^{1/r}} \ge \frac{|\mathbf{m}| \mathbf{E} Y_{\mathbf{m}}}{2^{d/r} |\mathbf{m}|^{1/r}} = \frac{\mathbf{E} Z_{\mathbf{m}}}{2^{d/r} |\mathbf{m}|^{1/r}},$$

so $|\mathbf{n}|^{-1/r} \mathbf{E} Z_{\mathbf{n}} \to 0$ as $|\mathbf{n}| \to \infty$. Hence 4) follows.

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