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A General Approach to Strong Laws of Large Numbers for Fields of Random Variables

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A GENERAL APPROACH TO STRONG LAWS OF LARGE NUMBERS FOR FIELDS OF RANDOM VARIABLES

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Abstract. A general method to prove strong laws of large numbers for random fields is presented. The method is applied for logarithmically weighted sums, fields with superadditive moment structure and mixingales.

1. Introduction And Notation

There are several methods to obtain almost sure (a.s.) convergence results for random fields (see e.g. [PG],[M],[K],[F] and the literature cited there). The aim of our paper is to present a general approach to obtain strong laws of large numbers (SLLN) for random fields. Our method is an extension of the one given in [FK]. In [FK] only random sequences (i.e. not fields) were considered.

The paper is organized as follows. In Section 2 a Hájek-Rényi type maximal inequality is given. Section 3 contains the main result (Theorem 3). Once a maximal inequality is known, Theorem 3 easily implies an SLLN and it helps to obtain appropriate normalizing constants in the SLLN. The remaining sections contain applications. In Section 4 an SLLN is presented for logarithmically weighted sums. We remark that such kind of SLLN's are useful to prove almost sure central limit theorems (see e.g. [CF]). In Section 5 the case of fields with superadditive moment structure is studied. In Section 6 a Brunk-Prokhorov type SLLN is presented. Section 7 is devoted to mixingales.

In the following $\mathbb{N}_0$ and $\mathbb{N}$ denote the set of nonnegative and positive integers, respectively. Let d be a fixed positive integer. Throughout the paper I, J, K, L, M and N denote elements of $\mathbb{N}_0^d$ (in particular, elements of $\mathbb{N}^d$). If an element of $\mathbb{N}_0^d$ (or $\mathbb{N}^d$) is denoted by a capital letter, then its coordinates are denoted by the lower case of the same letter, i.e. N always means the vector $(n_1, \dots, n_d) \in \mathbb{N}_0^d$. We also use $\mathbf{1} = (1, \dots, 1) \in \mathbb{N}^d$ and $\mathbf{0} = (0, \dots, 0) \in \mathbb{N}^d$. In $\mathbb{N}_0^d$ we consider the coordinate-wise partial ordering: $M \le N$ means $m_i \le n_i,\ i = 1, \dots, d$ ($M < N$ means $M \le N$ and $N \ne M$). $N \to \infty$ is interpreted as $n_i \to \infty,\ i = 1, \dots, d$, $\lim_N a_N$ is meant in this sense. In $\mathbb{N}_0^d$ the maximum is defined coordinate-wise (actually we shall use it only for rectangles). If $N = (n_1, \dots, n_d) \in \mathbb{N}_0^d$ then $\langle N \rangle = \prod_{i=1}^d n_i$.

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A numerical sequence $a_N,\ N\in\mathbb{N}^d$ is called d-sequence. If a_N is a d-sequence then its difference sequence, i.e. the d-sequence b_N for which $\sum_{M\le N} b_M = a_N,\ N\in\mathbb{N}^d$, will be denoted by $\Delta a_N (i.e. \Delta a_N = b_N)$.

We shall say that a d-sequence a_N is of product type if $a_N=\prod_{i=1}^d a_{n_i}^{(i)}$, where $a_{n_i}^{(i)}\ \ (n_i=1,2,\dots)$ is a (single) sequence for each $i=1,\dots,d$. Our consideration will be confined to normalizing constants of product type: b_N will always denote $b_N=\prod_{i=1}^d b_{n_i}^{(i)}$, where $b_{n_i}^{(i)}\ \ n_i=1,2,\dots$ is a nondecreasing sequence of positive numbers for each $i=1,\dots,d$. In this case we shall say that b_N is a positive nondecreasing d-sequence of product type. Moreover, if for each $i=1,\dots,d$ the sequence $b_{n_i}^{(i)}$ is unbounded, then b_N is called positive, nondecreasing, unbounded d-sequence of product type.

The random field will be denoted by $X_N,\ N\in\mathbb{N}_0^d$. S_N is partial sum: $S_N=\sum_{M\le N} X_M$ for $N\in\mathbb{N}_0^d$. As X_N is a field with lattice indices we shall say that $X_N,\ N\in\mathbb{N}_0^d$ is a d-sequence of random variables (r.v.'s). Remark that a sum or a maximum over the empty set will be interpreted as zero (i.e. $\sum_{N\in\mathcal{H}} X_N = \max_{N\in\mathcal{H}} X_N = 0$ if $\mathcal{H}=\emptyset$). As usual, $\log^+(x)=\max\{1,\log(x)\},\ x>0$.

2. Preliminary Results

The proposition and lemma below are useful for proving Theorem 3. Proposition 1 and its proof are straightforward generalizations of Theorem 1.1 and its proof in [FK]. Note that there are several other ways to obtain maximal inequalities of this type: see for example [K].

Proposition 1. *(Hájek-Rényi type maximal inequality.) Let r be a positive real number, a_N be a nonnegative d-sequence. Suppose that b_N is a positive, nondecreasing d-sequence of product type. Then for any fixed $N\in\mathbb{N}^d$*

$$\mathbb{E}\Big\{\max_{L\le M}|S_L|^r\Big\}\le\sum_{L\le M} a_L\qquad \forall M\le N$$

implies

$$\mathbb{E}\Big\{\max_{M\le N}\Big|\frac{S_M}{b_M}\Big|^r\Big\}\le 4^d\sum_{M\le N}\frac{a_M}{b_M^r}.$$

Proof. Without loss of generality we can assume that $b_{(1,..,1)}=1$. Fix an $N\in\mathbb{N}^d$ and for a moment a real number $c>1$. For $I=(i_1,\dots,i_d)\in\mathbb{N}_0^d$ let us define the set

$$\mathcal{A}_I=\{\ J\in\mathbb{N}^d\ :\ J\le N \text{ and } c^{i_k}\le b_{j_k}^{(k)}<c^{i_k+1},\ k=1,\dots,d\ \}.$$

Now we can form

$$D_I=\sum_{J\in\mathcal{A}_I} a_J \ \text{ and }\ K=\max\{\ I\ :\ \mathcal{A}_I\neq\emptyset\ \},$$

where D_I as we mentioned above is considered to be zero if $\mathcal{A}_I = \emptyset$. Note that K is well defined because of product form of b_N. It is easy to see that each nonempty $\mathcal{A}_I$ has a maximal element. Therefore if $\mathcal{A}_I \neq \emptyset$ let

$$M_I = \max\{\ J\ :\ J \in \mathcal{A}_I\ \}$$

otherwise set $M_I = (0, .., 0)$. Since $\bigcup_{I \le K} \mathcal{A}_I$ covers the rectangle $\{M \in \mathbb{N}^d\ :\ M \le N\}$ so

$$\mathbb{E}\Big\{\max_{M \le N}\Big|\frac{S_M}{b_M}\Big|^r\Big\} \le \sum_{J \le K}\mathbb{E}\Big\{\max_{I \in A_J}\Big|\frac{S_I}{b_I}\Big|^r\Big\}.$$

By the definition of $\mathcal{A}_I$, M_I and D_I we get

$$\sum_{J \le K}\mathbb{E}\Big\{\max_{I \in A_J}\Big|\frac{S_I}{b_I}\Big|^r\Big\} \le \sum_{J \le K}\Big\{\prod_{m=1}^d c^{-rj_m}\Big\}\mathbb{E}\Big\{\max_{I \in \mathcal{A}_J}|S_I|^r\Big\} \le$$

$$\sum_{J \le K}\Big\{\prod_{m=1}^d c^{-rj_m}\Big\}\mathbb{E}\Big\{\max_{I \le M_J}|S_I|^r\Big\} \le \sum_{J \le K}\Big\{\prod_{m=1}^d c^{-rj_m}\Big\}\sum_{I \le M_J} a_I \le$$

$$\sum_{J \le K}\Big\{\prod_{m=1}^d c^{-rj_m}\Big\}\sum_{I \le J} D_I \le \sum_{I \le K} D_I \sum_{I \le J \le K}\Big\{\prod_{m=1}^d c^{-rj_m}\Big\} \le$$

$$\sum_{I \le K} D_I \prod_{m=1}^d\Big\{\sum_{j=i_m}^{k_m} c^{-rj}\Big\} \le \sum_{I \le K} D_I \prod_{m=1}^d \frac{c^{-ri_m}}{1 - c^{-r}} \le$$

$$\Big\{\frac{c^r}{1 - c^{-r}}\Big\}^d \sum_{I \le K} D_I \prod_{m=1}^d c^{-r(i_m+1)} \le$$

$$\Big\{\frac{c^r}{1 - c^{-r}}\Big\}^d \sum_{I \le K}\Big\{\sum_{J \in \mathcal{A}_I} a_J\Big\}\prod_{m=1}^d c^{-r(i_m+1)} \le$$

$$\Big\{\frac{c^r}{1 - c^{-r}}\Big\}^d \sum_{I \le K}\sum_{J \in \mathcal{A}_I}\frac{a_J}{b_J^r} \le \Big\{\frac{c^r}{1 - c^{-r}}\Big\}^d \sum_{J \le N}\frac{a_J}{b_J^r}.$$

This proves the proposition because $\inf_{c>1}\frac{c^r}{1-c^{-r}} = 4$. $\square$

Lemma 2. *Let a_N be a nonnegative d-sequence and let b_N be a positive, nondecreasing, unbounded d-sequence of product type. Suppose that $\sum_N \frac{a_N}{b_N^r} < +\infty$ with a fixed real $r > 0$. Then there exists a positive, nondecreasing, unbounded d-sequence β_N of product type for which*

$$\lim_N \frac{\beta_N}{b_N} = 0 \quad \textit{and} \quad \sum_N \frac{a_N}{\beta_N^r} < +\infty.$$

Proof. Clearly it is enough to prove for $r = 1$. In case of $d = 1$ one can find our proposition in [FK, Lemma 2.2]. Let $d \geq 2$. Then

$$+\infty > \sum_N \frac{a_N}{b_N} = \sum_{n_1} \frac{1}{b^{(1)}_{n_1}} \sum_{n_2,..,n_d} \frac{a_N}{\prod_{m=2}^d b^{(m)}_{n_m}} = \sum_{n_1} \frac{1}{b^{(1)}_{n_1}} T_{n_1}$$

with $T_{n_1} = \sum_{n_2,..,n_d} \frac{a_N}{\prod_{m=2}^d b^{(m)}_{n_m}}$. Applying the above mentioned lemma of [FK], we get that there exists an unbounded, positive, nondecreasing sequence $\beta^{(1)}_n$ so that

$$\lim_n \frac{\beta^{(1)}_n}{b^{(1)}_n} = 0 \quad \text{and} \quad \sum_{n_1} \frac{1}{\beta^{(1)}_{n_1}} T_{n_1} < +\infty.$$

If we have already obtained $\beta^{(m)}_n$ for $m = 1, \ldots, k, \quad k < d$ then replacing in the above procedure b_N by $\prod_{m=1}^k \beta^{(m)}_{n_m} \prod_{m=k+1}^d b^{(l)}_{n_l}$ and coordinate 1 by coordinate $k+1$ we get an appropriate $\beta^{(k+1)}_n$. Finally, by setting $\beta_N = \prod_{m=1}^d \beta^{(m)}_{n_m}$, it obviously satisfies the requirements. □

3. The Main Theorem

The following theorem is an extension of Theorem 2.1 of [FK].

Theorem 3. *Let a_N, b_N be nonnegative d-sequences and let $r > 0$. Suppose that b_N is a positive, nondecreasing, unbounded d-sequence of product type. Then*

$$\sum_N \frac{a_N}{b^r_N} < +\infty$$

and

$$\mathbb{E}\Big\{\max_{M \leq N} |S_M|^r\Big\} \leq \sum_{M \leq N} a_M \qquad \forall\, N \in \mathbb{N}^d$$

imply

$$\lim_N \frac{S_N}{b_N} = 0 \quad a.s.$$

Proof. Let β_N be the sequence obtained in the previous lemma. According to Proposition 1:

$$\mathbb{E}\Big\{\max_{M \leq N} \Big|\frac{S_M}{\beta_M}\Big|^r\Big\} \leq 4^d \sum_{M \leq N} \frac{a_M}{\beta^r_M} \quad \forall N \in \mathbb{N}^d.$$

Hence

$$\mathbb{E}\Big\{\sup_{n_d} \ldots \sup_{n_1} \Big|\frac{S_N}{\beta_N}\Big|^r\Big\} \leq 4^d \sum_N \frac{a_N}{\beta^r_N}.$$

Since

$$\sup_{n_d} \dots \sup_{n_1} \left|\frac{S_N}{\beta_N}\right|^r = \sup_N \left|\frac{S_N}{\beta_N}\right|^r$$

it follows from the foregoing that

$$\sup_N \left|\frac{S_N}{\beta_N}\right|^r < +\infty \quad \text{a.s.}$$

Therefore

$$\left|\frac{S_N}{b_N}\right| = \frac{\beta_N}{b_N}\left|\frac{S_N}{\beta_N}\right| \le \frac{\beta_N}{b_N} \sup_K \left|\frac{S_K}{\beta_K}\right|.$$

This proves the theorem because $\lim_N \frac{\beta_N}{b_N} = 0$. □

Remark 4. Maximal inequalities play important role in proving SLLN's. We shall frequently use the following result of Móricz (see [M, Corollary 1] or [Mo2, Theorem 7]):
Let $r \geq 1, \gamma > 1$ and let g be nonnegative, superadditive function on $\mathbb{N}^d \times \mathbb{N}^d$. Let X_N be a d-sequence of random variables such that for any $I, J \in \mathbb{N}^d$,

$$\mathbb{E}\Big\{\Big|\sum_{I\le K\le J} X_K\Big|^r\Big\} \le g(I,J)^\gamma.$$

Then there is a constant $A_{r,\gamma,d}$ for which

$$\mathbb{E}\Big\{\max_{K\le N} |S_K|^r\Big\} \le A_{r,\gamma,d}\ g(1,N)^\gamma \qquad \forall N \in \mathbb{N}^d.$$

The reader can easy verify that the above proposition is true in the case of $0 < r < 1$ too. – A similar (but more special) inequality can be obtained with induction from [LS, Lemma 2].
Remark that a function g on $\mathbb{N}^d \times \mathbb{N}^d$ is said to be superadditive if

$$g\big(I, (j_1, \dots, j_{m-1}, k, j_{m+1}, \dots, j_d)\big) + g\big(i_1, \dots, i_{m-1}, k+1, i_{m+1}, \dots, i_d), J\big)$$

can be majorized by $g(I,J)$ for any $m = 1, \dots, d$ and for any $i_m \le k < j_m$.

As an illustration for application of Theorem 3 we prove the following simple result:

Corollary 5. *Let b_N be a positive, nondecreasing, unbounded d-sequence of product type and let X_N be a d-sequence of random variables such that $\sum_N \frac{X_N}{b_N}$ is absolutely convergent in L^p for certain $p > 1$. Then $\lim_N \frac{S_N}{b_N} = 0$ a.s.*

Proof. By Remark 4 and the triangle inequality, there is a constant C_p not depending on N such that:

$$\mathbb{E}\Big\{\max_{M\le N} |S_M|^p\Big\} \le C_p\Big\{\sum_{M\le N} \|X_M\|_p\Big\}^p.$$

It follows that
$$\mathbb{E}\Big\{\max_{M\le N}|S_M|\Big\}\le C_p^{\frac{1}{p}}\Big\{\sum_{M\le N}\|X_M\|_p\Big\}.$$
Now we apply Theorem 3. $\square$

4. Logarithmically Weighted Sums

Móri in [TM, Theorem 1] proved that the sequence
$$\frac{1}{\log^+ n}\sum_{k=1}^{n}\frac{X_k}{k}$$
converges a.s. to zero under general assumptions. With their general method in [FK] Fazekas and Klesov proved a special case of Móri's theorem. Now we extend this case to fields of random variables. Our method is a generalization that of [FK]. Our Lemma 6 and Theorem 7 are extensions of Lemma 9.1 and Theorem 9.1 of [FK], respectively. Proposition 9 covers the case when the natural normalization is not logarithmic but a power function. Proposition 8 deals with the special case of orthogonal d-sequences.
In the lemma below $[x]$, $x\ge 0$ denotes the integer part of x, i.e. [x] is the largest integer for which $[x]\le x$.

Lemma 6. *(a) Let $n\in\mathbb{N}$ and $0<\beta<1$. Then there is a constant $C_{d,\beta}$ depending only d and β such that:*
$$\sum_{m_1=1}^{n}\sum_{m_2=1}^{\left[\frac{n}{m_1}\right]}\cdots\sum_{m_d=1}^{\left[\frac{n}{m_1m_2\cdots m_{d-1}}\right]}\frac{1}{\langle M\rangle^{1-\beta}}\le C_{d,\beta}\;n^{\beta}(\log^+ n)^{d-1}.$$
(b) Let $0<\beta<1$, $1<\gamma<2$, $I,M,J\in\mathbb{N}^d$, $I\le M\le J$. Then there is a constant $C_{d,\beta}$ depending only on d and β such that:
$$\sum_{I\le M\le J}\;\sum_{\substack{I\le K\le J\\ \langle K\rangle\le\langle M\rangle}}\frac{1}{\langle M\rangle^{1+\beta}}\frac{1}{(\log^+\langle M\rangle)^{d-1}}\frac{1}{\langle K\rangle^{1-\beta}}\le C_{d,\beta}\Big\{\sum_{I\le M\le J}\frac{1}{\langle M\rangle}\Big\}^{\gamma}.$$

Proof. (a) The case $d=1$ is well known from elementary analysis. We prove by induction on d. Suppose that the statement is true for $d=f$. Let $n\in\mathbb{N}$ and $0<\beta<1$. Then
$$\sum_{m_1=1}^{n}\sum_{m_2=1}^{\left[\frac{n}{m_1}\right]}\cdots\sum_{m_{f+1}=1}^{\left[\frac{n}{m_1m_2\cdots m_f}\right]}\frac{1}{\langle M\rangle^{1-\beta}}=$$
$$\sum_{m_1=1}^{n}\frac{1}{m_1^{1-\beta}}\sum_{m_2=1}^{\left[\frac{n}{m_1}\right]}\cdots\sum_{m_{f+1}=1}^{\left[\frac{n}{m_1m_2\cdots m_f}\right]}\frac{1}{(m_2\cdots m_f)^{1-\beta}}.$$

Now applying the hypothesis for $\left[\frac{n}{m_1}\right]$ we get that the above expression is majorized by:

$$C_{f,\beta}\sum_{m_1=1}^{n}\frac{1}{m_1^{1-\beta}}\left[\frac{n}{m_1}\right]^{\beta}\left\{\log^+\left[\frac{n}{m_1}\right]\right\}^{f-1}\le C_{f,\beta}\;n^{\beta}(\log^+ n)^{f-1}\sum_{m_1=1}^{n}\frac{1}{m_1}\le$$

$$C_{f,\beta}\;n^{\beta}(\log^+ n)^{f-1}C\log^+ n$$

with certain $C>0$ (here we used the fact $\left[\frac{1}{c}\left[\frac{a}{b}\right]\right]=\left[\frac{a}{bc}\right]$ for $a,b,c\in\mathbb{N}$).

(b) In case $\sum_{I\le M\le J}\frac{1}{\langle M\rangle}\le 1$ we get that

$$\sum_{I\le M\le J}\sum_{\substack{I\le K\le J\\ \langle K\rangle\le\langle M\rangle}}\frac{1}{\langle M\rangle^{1+\beta}(\log^+\langle M\rangle)^{d-1}\langle K\rangle^{1-\beta}}\le$$

$$\sum_{I\le M\le J}\sum_{\substack{I\le K\le J\\ \langle K\rangle\le\langle M\rangle}}\frac{1}{\langle M\rangle^{1+\beta}\langle K\rangle^{1-\beta}}\le$$

$$\sum_{I\le M\le J}\sum_{\substack{I\le K\le J\\ \langle K\rangle\le\langle M\rangle}}\frac{1}{\langle M\rangle\langle K\rangle}\le\left\{\sum_{I\le M\le J}\frac{1}{\langle M\rangle}\right\}^2\le\left\{\sum_{I\le M\le J}\frac{1}{\langle M\rangle}\right\}^{\gamma}.$$

In case $\sum_{I\le M\le J}\frac{1}{\langle M\rangle}>1$ using part (a) and the simple fact

$$\sum_{m_1=1}^{n}\sum_{m_2=1}^{\left[\frac{n}{m_1}\right]}\cdots\sum_{m_d=1}^{\left[\frac{n}{m_1m_2\cdots m_{d-1}}\right]}\frac{1}{\langle M\rangle^{1-\beta}}=\sum_{\substack{M\in\mathbb{N}^d\\ \langle M\rangle\le n}}\frac{1}{\langle M\rangle^{1-\beta}},\qquad\forall n,d\in\mathbb{N}$$

we get that

$$\sum_{I\le M\le J}\frac{1}{\langle M\rangle^{1+\beta}(\log^+\langle M\rangle)^{d-1}}\sum_{\substack{I\le K\le J\\ \langle K\rangle\le\langle M\rangle}}\frac{1}{\langle K\rangle^{1-\beta}}\le$$

$$C_{d,\beta}\sum_{I\le M\le J}\frac{1}{\langle M\rangle^{1+\beta}(\log^+\langle M\rangle)^{d-1}}\langle M\rangle^{\beta}(\log^+\langle M\rangle)^{d-1}=$$

$$C_{d,\beta}\sum_{I\le M\le J}\frac{1}{\langle M\rangle}\le C_{d,\beta}\left\{\sum_{I\le M\le J}\frac{1}{\langle M\rangle}\right\}^{\gamma}.\quad\square$$

Theorem 7. *Let $X_N, N\in\mathbb{N}^d$, be a d-sequence of random variables and suppose that for some $C>0,\ \beta>0$*

$$|\mathbb{E}(X_KX_L)|\le C\left\{\frac{\langle K\rangle}{\langle L\rangle}\right\}^{\beta}\frac{1}{(\log^+\langle L\rangle)^{d-1}}\quad\textit{if}\ \ \langle K\rangle\le\langle L\rangle,$$

where $\langle N\rangle = \prod_{i=1}^d n_i$. *Then*

$$\lim_N \frac{1}{\prod_{i=1}^d \log^+ n_i} \sum_{K\le N} \frac{X_K}{\langle K\rangle} = 0 \quad a.s.$$

Proof. Clearly it is enough to prove for $0<\beta<1$. Let $I, J \in \mathbb{N}^d,\ I \le J$. Using the assumptions we get:

$$\mathbb{E}\Big\{\Big|\sum_{I\le K\le J} \frac{X_K}{\langle K\rangle}\Big|^2\Big\} \le 2\sum_{I\le L\le J} \sum_{\substack{I\le K\le J\\ \langle K\rangle\le\langle L\rangle}} \frac{1}{\langle K\rangle\langle L\rangle} |\mathbb{E}(X_K X_L)| \le$$

$$2C \sum_{I\le L\le J} \sum_{\substack{I\le K\le J\\ \langle K\rangle\le\langle L\rangle}} \frac{1}{\langle K\rangle^{1-\beta}\langle L\rangle^{1+\beta}(\log^+\langle L\rangle)^{d-1}}.$$

Let $1<\gamma<2$. It follows from Lemma 6(b) that

$$\mathbb{E}\Big\{\Big|\sum_{I\le K\le J} \frac{X_K}{\langle K\rangle}\Big|^2\Big\} \le D_{d,\beta}\Big\{\sum_{I\le L\le J} \frac{1}{\langle L\rangle}\Big\}^\gamma,$$

where $D_{d,\beta}>0$ depends only on d and β. Now, from Remark 4 we get that

$$\mathbb{E}\Big\{\max_{I\le J}\Big|\sum_{K\le I} \frac{X_K}{\langle K\rangle}\Big|^2\Big\} \le C_{d,\beta,\gamma}\Big\{\sum_{K\le J} \frac{1}{\langle K\rangle}\Big\}^\gamma \qquad \forall J,$$

where $C_{d,\beta,\gamma}>0$ depends only on d, β and γ. ¿From the Hölder inequality we have:

$$\mathbb{E}\Big\{\max_{I\le J}\Big|\sum_{K\le I} \frac{X_K}{\langle K\rangle}\Big|^{\frac{2}{\gamma}}\Big\} \le (C_{d,\beta,\gamma})^{\frac{1}{\gamma}} \sum_{K\le J} \frac{1}{\langle K\rangle} \qquad \forall J.$$

Now we can apply Theorem 3 because

$$\sum_N \frac{1}{(\prod_{m=1}^d \log n_m)^{\frac{2}{\gamma}}} \frac{1}{\langle N\rangle} < +\infty. \qquad \square$$

Now we give some analogues of Theorem 7. First of all we consider the situation of orthogonal sequences.

Proposition 8. *Let X_N be an orthogonal d-sequence of random variables, $r>0$ and $s>\frac{1+r}{2}$. Suppose that for some $C>0$*

$$\mathbb{E}(X_K^2) \le C\langle K\rangle^r.$$

Then for any $\rho>1$

$$\lim_N \frac{1}{\big\{\prod_{i=1}^d \log^+ n_i\big\}^\rho} \sum_{K\le N} \frac{X_K}{\langle K\rangle^s} = 0 \quad a.s.$$

Proof. Using the d-multiple version of the Rademacher-Menšov inequality [M, Corollary 3a] and the assumptions, we have for a certain $C_d > 0$

$$\mathbb{E}\left\{\max_{M\le N}\left|\sum_{L\le M}\frac{X_L}{\langle L\rangle^s}\right|^2\right\} \le CC_d\left\{\prod_{i=1}^d \log^+(n_i+1)\right\}^2 \sum_{M\le N}\frac{1}{\langle M\rangle^{2s-r}}.$$

Let

$$B = CC_d\sum_N \frac{1}{\langle N\rangle^{2s-r}} \quad \text{and} \quad A_N = B\left\{\prod_{i=1}^d \log^+(n_i+1)\right\}^2.$$

Since $\{\log^+(n+1)\}^2 - \{\log^+ n\}^2 \le D\frac{\log^+ n}{n}$ for some constant D, so $A_N \le BD\sum_{M\le N}\frac{\prod_{i=1}^d \log^+ m_i}{\langle M\rangle}$, and therefore one can apply Theorem 3. □

Proposition 9. *Let $0 < r < 1$ and $0 < s \le \frac{2}{3-r}$. Suppose that for some $C > 0$*

$$|\mathbb{E}(X_K X_L)| \le \frac{C\langle K\rangle^{sr}}{\langle L\rangle^s(\log^+\langle L\rangle)^{d-1}} \quad \textit{if} \ \ \langle K\rangle \le \langle L\rangle.$$

Then

$$\lim_N \frac{1}{\langle N\rangle^{1-s}}\sum_{K\le N}\frac{X_K}{\langle K\rangle^s} = 0 \quad a.s.$$

Proof. The proof is similar to that of Theorem 7. Let $I, J \in \mathbb{N}$. Then

$$\mathbb{E}\left\{\left|\sum_{I\le K\le J}\frac{X_K}{\langle K\rangle^s}\right|^2\right\} \le 2C\sum_{I\le L\le J}\frac{1}{\langle L\rangle^{2s}(\log^+\langle L\rangle)^{d-1}}\sum_{\substack{I\le K\le J\\ \langle K\rangle\le\langle L\rangle}}\frac{1}{\langle K\rangle^{s(1-r)}} = (I)$$

By Lemma 6(a)

$$(I) \le C_{s,r,d}\sum_{I\le L\le J}\frac{1}{\langle L\rangle^{s(3-r)-1}}.$$

Let $1 < \gamma < 2$. If $(II) = \sum_{I\le L\le J}\frac{1}{\langle L\rangle^{s(3-r)-1}} \ge 1$ then $(II) \le (II)^\gamma$. If $(II) < 1$ then from $s \le \frac{2}{3-r}$ we get

$$(I) \le 2C\sum_{I\le L\le J}\sum_{\substack{I\le K\le J\\ \langle K\rangle\le\langle L\rangle}}\frac{1}{\langle L\rangle^{2s}\langle K\rangle^{s(1-r)}} \le$$

$$2C\sum_{I\le L\le J}\sum_{\substack{I\le K\le J\\ \langle K\rangle\le\langle L\rangle}}\frac{1}{\langle L\rangle^{s(3-r)-1}\langle K\rangle^{s(3-r)-1}} \le$$

$$2C\left\{\sum_{I\le L\le J}\frac{1}{\langle L\rangle^{s(3-r)-1}}\right\}^\gamma.$$

Using Remark 4, the above results and then Hölder's inequality we get

$$\mathbb{E}\left\{\max_{M\le N}\left|\sum_{L\le M}\frac{X_L}{\langle L\rangle^s}\right|^{\frac{2}{\gamma}}\right\}\le D\sum_{M\le N}\frac{1}{\langle M\rangle^{s(3-r)-1}},$$

where the constant D does not depend on N or γ. Considering the quantity $s(3-r)-1+\frac{2}{\gamma}(1-s)$, it is easy to see that it is greater than 1, for a suitable γ. Therefore the assumptions of Theorem 3 are satisfied. □

5. Sequences with superadditive moment structure

A d-sequence of random variables is said to have r-th moment function of superadditive structure ($MFSS$) if

$$\mathbb{E}\left\{\left|\sum_{I\le K\le J}X_K\right|^r\right\}\le g(I,J)^\alpha\quad\forall I,J\in\mathbb{N}^d,$$

where g is superadditive on $\mathbb{N}^d\times\mathbb{N}^d$, $r>0$ and $\alpha>1$. Remark that the notion of r-th $MFSS$ was used by Móricz in [Mo]. In this section we prove a Marcinkiewicz-Zygmund type SLLN for d-sequences with superadditive moment structure. Our Proposition 11 is a generalization of Theorem 8.1 of [FK]. For the sake of completness we start with a simple technical lemma on partial summation.

Lemma 10. *Let $a_N,\ b_N$ be nonnegative d-sequences such that $b_N=\frac{1}{\langle N\rangle^\alpha}$ for some $\alpha>0$. Then*

$$\sum_N(-1)^d\Lambda_N\Delta b_{N+\mathbf{1}}<+\infty$$

implies

$$\sum_N a_Nb_N<+\infty,$$

where $\Lambda_N=\sum_{M\le N}a_M$.

Proof. Set $b_N=0$ if $n_k<1$ for some $k=1,\dots,d$. Then

$$\Delta b_N=\sum_{M\in\mathcal{E}_N}b_M+(-1)\sum_{M\in\mathcal{O}_N}b_M,$$

where

$$\mathcal{E}_N=\left\{M\in\mathbb{N}_0^d\ :\ 0\le n_k-m_k\le 1,\quad k=1,\dots,d\ \text{ and }\ \sum_{k=1}^d(n_k-m_k)\ \text{ is even}\right\},$$

$$\mathcal{O}_N=\left\{M\in\mathbb{N}_0^d\ :\ 0\le n_k-m_k\le 1,\quad k=1,\dots,d\ \text{ and }\ \sum_{k=1}^d(n_k-m_k)\ \text{ is odd}\right\}.$$

It follows that $(-1)^d \Delta b_{N+\mathbf{1}} = \prod_{k=1}^d \Big\{ \frac{1}{n_k^\alpha} - \frac{1}{(n_k+1)^\alpha} \Big\}$. Since $\frac{1}{n^\alpha} - \frac{1}{(n+1)^\alpha} \geq C \frac{1}{n^{\alpha+1}}$ for some $C > 0$ so

$$\sum_N \frac{\Lambda_N}{\langle N \rangle^{\alpha+1}} < +\infty.$$

An elementary computation shows that

$$\sum_{K \leq 2N} \frac{\Lambda_K}{\langle K \rangle^{\alpha+1}} \geq \sum_{K \leq N} a_K \sum_{K \leq L \leq 2K} \frac{1}{\langle L \rangle^{\alpha+1}} \geq$$

$$\sum_{K \leq N} a_K \frac{\langle K \rangle}{\langle 2K \rangle^{\alpha+1}} = \frac{1}{2^{d(\alpha+1)}} \sum_{K \leq N} \frac{a_K}{\langle K \rangle^\alpha}.$$

This shows that $\sum_N a_N b_N < +\infty$. $\square$

Proposition 11. *Let $r > 0$ and suppose that X_N has r-th MFSS and $\Delta g(\mathbf{1}, N)^\alpha$ is nonnegative for any $N \in \mathbb{N}^d$. Then for arbitrary $q > 0$*

$$\sum_N \frac{g(\mathbf{1}, N)^\alpha}{\langle N \rangle^{1+\frac{r}{q}}} < +\infty \tag{I}$$

implies

$$\lim \frac{S_N}{\langle N \rangle^{\frac{1}{q}}} = 0 \quad a.s.$$

Proof. Using Remark 4 we get for all $N \in \mathbb{N}^d$ that:

$$\mathbb{E}\Big\{ \max_{M \leq N} |S_M|^r \Big\} \leq A_{d,r,\alpha}\, g(\mathbf{1}, N)^\alpha.$$

Let us introduce the notation $b_N = \frac{1}{\langle N \rangle^{\frac{r}{q}}}$. Since $\prod_{m=1}^d \Big\{ \frac{1}{n_m^{\frac{r}{q}}} - \frac{1}{(n_m+1)^{\frac{r}{q}}} \Big\} \leq C \frac{1}{\langle N \rangle^{1+\frac{r}{q}}}$ for some $C > 0$, so (I) implies

$$\sum_N (-1)^d g(\mathbf{1}, N)^\alpha \Delta b_{N+\mathbf{1}} < +\infty.$$

Finally, we apply Lemma 10 and Theorem 3 to obtain the result. $\square$

6. A Brunk-Prokhorov Type Theorem

Let $(\Omega, \mathcal{A}, P)$ be a probability space. Let X_N and $\mathcal{A}_N$ be a d-sequence of random variables and be a d-sequence of σ-subalgebras of $\mathcal{A}$, respectively. We shall say that the pair $(X_N, \mathcal{A}_N)$ has property (ex) if

$$\mathbb{E}\Big(\mathbb{E}(X_L|\mathcal{A}_M)|\mathcal{A}_N)\Big) = \mathbb{E}\Big(X_L|\mathcal{A}_{\min(M,N)}\Big) \qquad L, M, N \in \mathbb{N}^d. \tag{ex}$$

This property is widely used in the theory of multiindex martingales (see e.g. [F]). Let X_N be a d-sequence of random variables and $\mathcal{A}_N$ a nondecreasing d-sequence of sub σ-algebras of $\mathcal{A}$. We say that X_N is a martingale difference if

$$X_N \text{ is measurable with respect to } \mathcal{A}_N, \quad N \in \mathbb{N}^d,$$

$$\mathbb{E}(X_{\mathbf{1}}) = 0 \text{ and } \mathbb{E}(X_N \mid \mathcal{A}_M) = 0 \text{ if } M < N.$$

In this section we shall use the Khintchine, Doob and Burkholder inequalities for d-sequences of random variables. For the sake of completeness we state and prove these inequalities in the lemma below.

Lemma 12. *(a) (Doob's L^p-inequality.) Let $p > 1$. Then for any martingale $(X_N, \mathcal{A}_N)$ having property (ex) for arbitrary $N \in \mathbb{N}^d$*

$$\mathbb{E}\Big\{\max_{M \le N} |X_M|^p\Big\} \le \Big\{\frac{p}{p-1}\Big\}^{pd} \mathbb{E}(|X_N|^p).$$

(b) (Khintchine's inequality.) Let $p \ge 2$. Then there exists a constant $C_{p,d}$ (depending only on p and d), such that for arbitrary d-sequence a_N of real numbers and for any $N \in \mathbb{N}^d$

$$\int_0^1 \cdots \int_0^1 \Big| \sum_{m_1=1}^{n_1} \cdots \sum_{m_d=1}^{n_d} r_{m_1}(t_1) \cdots r_{m_d}(t_d) a_{(m_1,\ldots,m_d)} \Big|^p dt_1 \cdots dt_d \le$$

$$C_{p,d} \Big\{ \sum_{m_1=1}^{n_1} \cdots \sum_{m_d=1}^{n_d} a^2_{(m_1,\ldots,m_d)} \Big\}^{\frac{p}{2}},$$

where r_n is the Rademacher system on $[0,1]$.

(c) (Burkholder's inequality) Let $p > 1$. Then there is a constant $D_{p,d}$ such that for any martingale difference X_N having property (ex)

$$\mathbb{E}(|S_N|^{2p}) \le D_{p,d} \mathbb{E}\Big(\Big\{\sum_{M \le N} X_M^2\Big\}^p\Big) \qquad \forall N \in \mathbb{N}^d.$$

Proof. (a) We shall prove by induction on d. In case $d = 1$ the statement follows from Doob's original inequality. First suppose that the implication is true for $d = e$. Now let $N = (n_1, \ldots, n_{e+1}) \in \mathbb{N}^{e+1}$ be fixed and define

$$N^* = (n_2, \ldots, n_{e+1}) \in \mathbb{N}^e, \quad Y_n = \max_{M \le N^*} |X_{(n,M)}| \text{ and } \mathcal{B}_n = \mathcal{A}_{(n,N^*)}$$

for $1 \le n \le n_1$. Then Y_n is $\mathcal{B}_n$ measurable for $n \le n_1$. We shall prove that $(Y_n, \mathcal{B}_n)$, $n \le n_1$, is a submartingale. Let $m \le n \le n_1$. By elementary properties of the conditional expectation and property (ex):

$$\mathbb{E}\Big(\max_{M \le N^*} |X_{(n,M)}| \mid \mathcal{A}_{(m,M)}\Big) \ge \max_{M \le N^*} \mathbb{E}\Big(|X_{(n,M)}| \mid \mathcal{A}_{(m,M)}\Big) \ge$$

$$\max_{M\le N^*}\Big|\mathbb{E}\Big(X_{(n,M)} \mid \mathcal{A}_{(m,N^*)}\Big)\Big| = \max_{M\le N^*}\Big|\mathbb{E}\Big(\mathbb{E}\big(X_{(n,M)} \mid \mathcal{A}_{(n,M)}\big) \mid \mathcal{A}_{(m,N^*)}\Big)\Big| =$$

$$= \max_{M\le N^*}\Big|\mathbb{E}\Big(X_{(n,M)} \mid \mathcal{A}_{(m,M)}\Big)\Big| = \max_{M\le N^*}|X_{(m,M)}|.$$

For $M \in \mathbb{N}^e, M \le N^*$ define

$$Z_M = X_{(n_1,M)} \quad \text{and} \quad \mathcal{C}_M = \mathcal{A}_{(n_1,M)}.$$

Then $(Z_M, \mathcal{C}_M),\ M \le N^*$, is a martingale having property (ex). Now using induction and Doob's inequality for the nonnegative submartingale Y_n, we get:

$$\mathbb{E}\Big\{\max_{M\le N}|X_M|^p\Big\} = \mathbb{E}\Big\{\max_{m\le n_1} Y_m^p\Big\} \le$$

$$\Big\{\frac{p}{p-1}\Big\}\mathbb{E}\Big(Y_{n_1}^p\Big) = \Big\{\frac{p}{p-1}\Big\}\mathbb{E}\Big\{\max_{M\le N^*}|Z_M|^p\Big\} \le$$

$$\Big\{\frac{p}{p-1}\Big\}\Big\{\frac{p}{p-1}\Big\}^{pe}\mathbb{E}\Big(|Z_{N^*}|^p\Big) =$$

$$\Big\{\frac{p}{p-1}\Big\}^{p(e+1)}\mathbb{E}\Big(|X_N|^p\Big).$$

(b) Note that in the following we use frequently Fubini's theorem without explicitly mentioning it. Suppose that the statement is true for $d = f$. Then by Khintchine's original inequality [CT] there is a constant $C_p > 0$ for which

$$\int_0^1 \Big|\sum_{m_1=1}^{n_1} r_{m_1}(t_1)\Big\{\sum_{m_2=1}^{n_2}\cdots\sum_{m_{f+1}=1}^{n_{f+1}} r_{m_2}(t_2)\cdots r_{m_{f+1}}(t_{f+1})a_{(m_1,\dots,m_{f+1})}\Big\}\Big|^p dt_1 \le$$

$$C_p\Big\{\sum_{m_1=1}^{n_1}\Big\{\sum_{m_2=1}^{n_2}\cdots\sum_{m_{f+1}=1}^{n_{f+1}} r_{m_2}(t_2)\cdots r_{m_{f+1}}(t_{f+1})a_{(m_1,\dots,m_{f+1})}\Big\}^2\Big\}^{\frac{p}{2}} =$$

$$C_p\Big\{\sum_{m_1=1}^{n_1}\Big(I(t_2,\dots,t_{f+1})\Big)^2\Big\}^{\frac{p}{2}},$$

where $I(t_2,\dots,t_{f+1}) = \sum_{m_2=1}^{n_2}\cdots\sum_{m_{f+1}=1}^{n_{f+1}} r_{m_2}(t_2)\dots r_{m_{f+1}}(t_{f+1})a_{(m_1,\dots,m_{f+1})}$. By the triangle inequality in the space $L_{\frac{p}{2}}$

$$C_p\int_0^1\cdots\int_0^1\Big\{\sum_{m_1=1}^{n_1}\Big(I(t_2,\dots,t_{f+1})\Big)^2\Big\}^{\frac{p}{2}} dt_2\cdots dt_{f+1} \le$$

$$C_p\Big\{\sum_{m_1=1}^{n_1}\Big\{\int_0^1\cdots\int_0^1 |I(t_2,\dots,t_{f+1})|^p dt_2\cdots dt_{f+1}\Big\}^{\frac{2}{p}}\Big\}^{\frac{p}{2}}.$$

By induction, the expression above is majorized by

$$C_p C_{p,f} \left\{ \sum_{m_1=1}^{n_1} \sum_{m_2=1}^{n_2} \cdots \sum_{m_2=1}^{n_2} a^2_{(m_1,\ldots,m_{f+1})} \right\}^{\frac{p}{2}}.$$

(c) We follow the method of [Mx, Theorem 1]. Let $u^{(i)} : \mathbb{N} \to [0,1]$ for $i = 1, \ldots, d$ and $S_N = \sum_{M \le N} X_M$. Consider the sum

$$T_N = \sum_{M \le N} u^{(1)}_{m_1} \cdots u^{(d)}_{m_d} \; X_M =$$

$$\sum_{m_1=1}^{n_1} u^{(1)}_{m_1} \left\{ \sum_{m_2=1}^{n_2} \cdots \sum_{m_d=1}^{n_d} u^{(2)}_{m_2} \cdots u^{(d)}_{m_d} X_{(m_1,\ldots,m_d)} \right\} = \sum_{m_1=1}^{n_1} u^{(1)}_{m_1} Y_{m_1}.$$

Our first aim is to prove $\mathbb{E}(|T_N|^p) \le M_{p,d}\mathbb{E}(|S_N|^p)$ for a certain positive (depending only on p or d) constant $M_{p,d}$. In case $d = 1$ one can find this proposition in the proof of [B, Theorem 9]. We prove again with induction on d. Using property (ex) of X_N it is easy to check that Y_m is a martingale difference. Hence

$$\mathbb{E}(|T_N|^p) \le M_{p,1}\mathbb{E}\left\{ \left| \sum_{m_1=1}^{n_1} Y_{m_1} \right|^p \right\} =$$

$$M_{p,1}\mathbb{E}\left\{ \left| \sum_{m_2=1}^{n_2} \cdots \sum_{m_d=1}^{n_d} u^{(2)}_{m_2} \ldots u^{(d)}_{m_d} \left\{ \sum_{m_1=1}^{n_1} X_{(m_1,\ldots,m_d)} \right\} \right\} \right|^p.$$

Using property (ex) once again, it turns out that

$$Z_{(m_2,\ldots,m_d)} = \sum_{m_1=1}^{n_1} X_{(m_1,\ldots,m_d)}$$

is a martingale difference. Hence from the induction hypothesis we get that

$$\mathbb{E}(|T_N|^p) \le M_{p,1}M_{p,d-1}\mathbb{E}(|S_N|^p) = M_{p,d}\mathbb{E}(|S_N|^p).$$

Now from the foregoing, Fubini's theorem and Lemma 12(b) we get that $\mathbb{E}(|S_N|^{2p})$ can be majorized with

$$M_{2p,d} \int_0^1 \cdots \int_0^1 \mathbb{E}\left\{ \left| \sum_{m_1=1}^{n_1} \cdots \sum_{m_d=1}^{n_d} r_{m_1}(t_1) \ldots r_{m_d}(t_d) X_{(m_1,\ldots,m_d)} \right|^{2p} \right\} dt_1 \cdots dt_d =$$

$$M_{2p,d} \; \mathbb{E}\left\{ \int_0^1 \cdots \int_0^1 \left| \sum_{m_1=1}^{n_1} \cdots \sum_{m_d=1}^{n_d} r_{m_1}(t_1) \cdots r_{m_d}(t_d) X_{(m_1,\ldots,m_d)} \right|^{2p} dt_1 \cdots dt_d \right\} \le$$

$$M_{2p,d} \; C_{2p,d} \; \mathbb{E}\left\{ \left\{ \sum_{m_1=1}^{n_1} \cdots \sum_{m_d=1}^{n_d} X^2_{(m_1,\ldots,m_d)} \right\}^p \right\}. \qquad \square$$

Proposition 13. *Let X_N be a martingale difference having property (ex) and $p \geq 1$. Suppose that $\sum_{M\leq N} E(|X_M|^{2p}) \leq C\langle N\rangle^r$ for some $C>0$ and $r<p+1$. Then $\lim_N \frac{S_N}{\langle N\rangle} = 0$ a.s.*

Proof. From Burkholder's inequality (Lemma 12(c)) and Hölder's inequality

$$\mathbb{E}\big(|S_N|^{2p}\big) \leq D_{2p,2}\mathbb{E}\bigg\{\Big\{\sum_{M\leq N} X_M^2\Big\}^p\bigg\} \leq$$

$$D_{2p,2}\langle N\rangle^{p-1}\sum_{M\leq N}\mathbb{E}\big(|X_M|^{2p}\big) \leq D_{2p,2}\langle N\rangle^{p+r-1}.$$

Thus, by Doob's inequality (Lemma 12(a)),

$$\mathbb{E}\Big\{\max_{M\leq N}|S_M|^{2p}\Big\} \leq F_{2p,2}\sum_{M\leq N}\Delta\langle M\rangle^{p+r-1}$$

for some constant $F_{2p,2}>0$. Now $\Delta\langle M\rangle^{p+r-1} \leq C\langle M\rangle^{p+r-2}$ and Theorem 3 implies the result. $\square$

Proposition 14. *Let X_N be a martingale difference having property (ex) and let $p \geq 1$. Suppose that $\mathbb{E}(|X_N|^{2p})$ is d-sequence of product type. Then*

$$\sum_N \frac{\mathbb{E}(|X_N|^{2p})}{b_N^{2p}}\langle N\rangle^{p-1} < +\infty$$

implies $\lim_N \frac{S_N}{b_N} = 0$ a.s., provided that b_N is a nondecreasing, positive, unbounded d-sequence of product type and either $p=1$ or $\frac{\langle N\rangle^\delta}{b_N}$ is nonincreasing for some $\delta > \frac{p-1}{2p}$.

Proof. Applying Lemma 12(c), Hölder's inequality and Lemma 12(a) we get

$$\mathbb{E}\Big\{\max_{M\leq N}|S_M|^{2p}\Big\} \leq C_{p,d}\langle N\rangle^{p-1}\sum_{M\leq N}\mathbb{E}(|X_M|^{2p})$$

for some $C_{p,d}>0$. In case $p=1$ our main theorem and the above inequality imply the result. Let $p>1$. Introduce the notation $c_N = \langle N\rangle^{p-1}\sum_{M\leq N}\mathbb{E}(|X_M|^{2p})$. It is easy to see that

$$\Delta c_N = \prod_{l=1}^d\left\{n_l^{p-1}\sum_{k=1}^{n_l}a_k^{(l)} - (n_l-1)^{p-1}\sum_{k=1}^{n_l-1}a_k^{(l)}\right\} =$$

$$\prod_{l=1}^d\left\{n_l^{p-1}a_{n_l}^{(l)} + \{n_l^{p-1}-(n_l-1)^{p-1}\}\sum_{k=1}^{n_l-1}a_k^{(l)}\right\} \leq \prod_{l=1}^d\left\{n_l^{p-1}a_{n_l}^{(l)} + Cn_l^{p-2}\sum_{k=1}^{n_l-1}a_k^{(l)}\right\}$$

for some $C > 0$, where $\prod_{l=1}^{d} a_{n_l}^{(l)} = \mathbb{E}(|X_N|^{2p})$. Using the assumptions we get

$$\sum_{m=1}^{n} \frac{m^{p-2}}{b_m^{(l)2p}} \sum_{k=1}^{m-1} a_k^{(l)} = \sum_{k=1}^{n-1} a_k^{(l)} \sum_{m=k+1}^{n} \frac{m^{p-2}}{b_m^{(l)2p}} \leq \sum_{k=1}^{n} a_k^{(l)} \sum_{m=k}^{\infty} \frac{m^{p-2}}{b_m^{(l)2p}} =$$

$$\sum_{k=1}^{n} a_k^{(l)} \sum_{m=k}^{\infty} \frac{1}{m^r} \frac{m^{p+r-2}}{b_m^{(l)2p}} \leq \sum_{k=1}^{n} a_k^{(l)} \frac{k^{p+r-2}}{b_k^{(l)2p}} \sum_{m=k}^{\infty} \frac{1}{m^r} \leq \sum_{k=1}^{n} a_k^{(l)} \frac{k^{p+r-2}}{b_k^{(l)2p}} C k^{1-r}$$

for some $r > 1, C_r > 0$ and for each $1 \leq l \leq d$. This means that $\sum_N \Delta \frac{c_N}{b_N^{2p}} < +\infty$, hence one can apply Theorem 3. $\square$

We remark that a similar proposition can be proved in a similar manner for d-sequences having maximal coefficient of correlation strictly smaller than 1. For this, one can use [PG, Lemma 4] instead of Burkholder's inequality.

7. Mixingales

In this chapter we define multiindex L^r mixingales and prove an SLLN for a special class of such mixingales. Remark that the notion of L^r mixingales was introduced by McLeish [McL] and Andrews [A]. Let $\mathbb{Z}$ denote the set of integers and let

$$\mathcal{E}_N = \left\{ M \in \mathbb{Z}^d \ : \ 0 \leq n_k - m_k \leq 1 \quad k = 1, \ldots, d \ \text{ and } \ \sum_{k=1}^{d} (n_k - m_k) \ \text{ is even} \right\},$$

$$\mathcal{O}_N = \left\{ M \in \mathbb{Z}^d \ : \ 0 \leq n_k - m_k \leq 1 \quad k = 1, \ldots, d \ \text{ and } \ \sum_{k=1}^{d} (n_k - m_k) \ \text{ is odd} \right\},$$

if $N \in \mathbb{Z}^d$.

Definition 15. *Let $r \geq 1$, $(\Omega, \mathcal{A}, P)$ be a probability space, X_N be a d-sequence of random variables with finite r-th moment, $\mathcal{A}_N$ $(N \in \mathbb{Z}^d)$ be a nondecreasing d-sequence of σ-subalgebras of $\mathcal{A}$. The pair $(X_N, \mathcal{A}_M)$ $(N \in \mathbb{N}^d, M \in \mathbb{Z}^d)$ is called L^r-mixingale if*

(a) $$\left\| \mathbb{E}(X_N | \mathcal{A}_{N-M}) \right\|_r \leq c_N \Psi_{-M} \quad \textit{if} \ \ m_i \geq 0 \ \ \textit{for some} \ \ i = 1, \ldots, d,$$

(b) $$\left\| X_N - \mathbb{E}(X_N | \mathcal{A}_{N+M}) \right\|_r \leq c_N \Psi_M \quad \textit{if} \ \ M \geq \mathbf{0},$$

where c_N $(N \in \mathbb{N}^d), \Psi_N$ $(N \in \mathbb{Z}^d)$ are d-sequences with $\Psi_N \to 0$ as $n_i \to -\infty$ for some $i = 1, \ldots, d$, $\Psi_N \to 0$ as $n_i \to \infty$ for each $i = 1, \ldots, d$, and there is a constant $C > 0$ for which

$$\Psi_M \leq C \Psi_N$$

for any $N \in \mathbb{Z}^d$ and $M \in \mathcal{E}_N \cup \mathcal{O}_N$.

The following lemma is a straightforward generalization of Lemma 1 and Lemma 2 of [H].

Lemma 16. *(a) Let $r \geq 2$ and $(X_N, \mathcal{A}_M)$ $(N \in \mathbb{N}^d, M \in \mathbb{Z}^d)$ be an L^r mixingale, having property (ex). Then there exists an $F_{r,d} > 0$ such that*

$$\Big\| \max_{M \leq N} |S_M| \Big\|_r \leq F_{r,d} \sum_{K \in \mathbb{Z}^d} \left\{ \sum_{M \leq N} \left\| X_M^{(K)} \right\|_r^2 \right\}^{\frac{1}{2}},$$

where $X_M^{(K)} = \Delta \mathbb{E}(X_M | \mathcal{A}_{M-K})$ and here the difference is taken according to the subscript of $\mathcal{A}$ while the subscript of X remains fixed.

(b) Let $r \geq 2$ and $(X_N, \mathcal{A}_M)$ $(N \in \mathbb{N}^d, M \in \mathbb{Z}^d)$ be an L^r mixingale, having property (ex) such that $\sum_{K \in \mathbb{Z}^d} \Psi_K < +\infty$. Then

$$\Big\| \max_{M \leq N} |S_M| \Big\|_r \leq C_{r,d} \left\{ \sum_{M \leq N} c_M^2 \right\}^{\frac{1}{2}}$$

for some $C_{r,d}$.

Proof. (a) Let $N, K \in \mathbb{N}^d$. Then

$$\sum_{-K \leq M \leq K} X_N^{(M)} = \sum_{-K \leq M \leq K} \Delta \mathbb{E}(X_N | \mathcal{A}_{N-M}) =$$

$$\mathbb{E}(X_N | \mathcal{A}_{N+K}) + \sum_{L \in \mathcal{L}_K^+} \mathbb{E}(X_N | \mathcal{A}_L) + (-1) \sum_{L \in \mathcal{L}_K^-} \mathbb{E}(X_N | \mathcal{A}_L),$$

where

$$\mathcal{L}_K^+ = \{ L \in \mathbb{Z}^d \ : \ l_i = k_i \text{ if } i \notin I \text{ and } \ l_i = -(k_i + 1) \text{ if } \ i \in I,$$
$$\text{for some } I \subset \{1, \dots, d\}, \text{ with } I \neq \emptyset \text{ and } \operatorname{card}(I) \text{ is even } \},$$

$$\mathcal{L}_K^- = \{ L \in \mathbb{Z}^d \ : \ l_i = k_i \text{ if } i \notin I \text{ and } \ l_i = -(k_i + 1) \text{ if } \ i \in I,$$
$$\text{for some } I \subset \{1, \dots, d\}, \text{ with } \operatorname{card}(I) \text{ is odd } \}.$$

By the definition of the L^r-mixingale, one can see that

$$\lim_K \left\{ \sum_{-K \leq M \leq K} X_N^{(M)} - \mathbb{E}(X_N | \mathcal{A}_{N+K}) \right\} = \mathbf{0} \quad \text{in } L^r$$

and so

$$\lim_K \left\{ \sum_{-K \leq M \leq K} X_N^{(M)} - X_N \right\} = \mathbf{0} \quad \text{in } L^r \ .$$

Hence, using the triangle inequality in L^r, we get

$$\Big\| \max_{M \leq N} |S_M| \Big\|_r = \Big\| \max_{M \leq N} \Big| \sum_{L \leq M} \sum_{K \in Z^d} X_L^{(K)} \Big| \ \Big\|_r \leq$$

$$\Big\| \max_{M\le N} \sum_{K\in Z^d} \Big| \sum_{L\le M} X_L^{(K)} \Big| \Big\|_r \le \sum_{K\in Z^d} \Big\| \max_{M\le N} \Big| \sum_{L\le M} X_L^{(K)} \Big| \Big\|_r = (I).$$

Let $K \in \mathbb{N}^d$ be fixed. With the help of property (ex) it is easy to check that the pair $(Z_M, \mathcal{F}_M)$ is martingale difference, where

$$Z_M = X_M^{(K)} \quad \text{and} \quad \mathcal{F}_M = \mathcal{A}_{M-K}.$$

Hence by Lemma 12 (a), (c) and by the triangle inequality in the space $L^{\frac{r}{2}}$, we have

$$(I) \le D_{r,d} \sum_{K\in Z^d} \Big\| \sum_{L\le N} X_L^{(K)} \Big\|_r \le F_{r,d} \sum_{K\in Z^d} \Big\| \Big\{ \sum_{L\le N} \Big| X_L^{(K)} \Big|^2 \Big\}^{\frac{1}{2}} \Big\|_r =$$

$$F_{r,d} \sum_{K\in Z^d} \Big\| \Big\{ \sum_{L\le N} \Big| X_L^{(K)} \Big|^2 \Big\} \Big\|_{\frac{r}{2}}^{\frac{1}{2}} = F_{r,d} \sum_{K\in Z^d} \Big\{ \sum_{L\le N} \Big\| \Big| X_L^{(K)} \Big|^2 \Big\|_{\frac{r}{2}} \Big\}^{\frac{1}{2}}.$$

(b) Let us consider $X_L^{(K)}$. If $k_m \ge 0$ for some $m = 1, \dots, d$ then

$$\Big\| \Delta\mathbb{E}(X_L | \mathcal{A}_{L-K}) \Big\|_r \le c_L 2^d C \Psi_{-K}.$$

Otherwise, if $k_m \le -1$ for each $m = 1, \dots, d$, then by Definition 15,

$$\Big\| \Delta\mathbb{E}(X_L | \mathcal{A}_{L-K}) \Big\|_r \le \sum_{M\in \mathcal{E}_{L-K}\cup\mathcal{O}_{L-K}} \Big\| X_L - \mathbb{E}(X_L|\mathcal{A}_M) \Big\|_r \le c_L 2^d C \Psi_{-K}.$$

Hence, by part (a),

$$\Big\| \max_{M\le N} |S_M| \Big\|_r \le F_{r,d} \sum_{K\in Z^d} \Big\{ \sum_{L\le N} c_L^2 2^{2d} C^2 \Psi_{-K}^2 \Big\}^{\frac{1}{2}} =$$

$$F_{r,d} 2^d C \Big\{ \sum_{K\in Z^d} \Psi_K \Big\} \Big\{ \sum_{L\le N} c_L^2 \Big\}^{\frac{1}{2}}. \qquad \square$$

Proposition 17. *Let $r \ge 2$ and $(X_N, \mathcal{A}_M)$ $(N \in \mathbb{N}^d, M \in \mathbb{Z}^d)$ be an L^r mixingale of property (ex). Then*

$$\sum_{N\in\mathbb{Z}^d} \Psi_N < \infty \quad \text{and} \quad \sum_{N\in\mathbb{N}^d} \frac{1}{\langle N\rangle^{1+\frac{r}{q}}} \Big\{ \sum_{M\le N} c_M^2 \Big\}^{\frac{r}{2}} < \infty$$

imply

$$\lim_N \frac{S_N}{\langle N\rangle^{\frac{1}{q}}} = 0 \quad a.s.$$

provided that the d-sequence c_N is of product type.

Proof. Easy consequence of Proposition 11 and Lemma 16(b). $\square$

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